Chapter 3

Linear Transformations, Eigenvectors and Similarity

3.1 Changes of Basis

The dimension of the space $\mathbb{F}^n$ is n - the *standard basis* consists of the column vectors $e_1, \dots, e_n$, where e_i has 1 in position i and zeros in all other positions.

For example, in $\mathbb{R}^3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the standard basis $\mathcal{E} = \{e_1, e_2, e_3\}$.

3.1.1 How can we recognize a basis of $\mathbb{F}^n$?

It should have n elements, which should be column vectors in $\mathbb{F}^n$. But some sets of three column vectors in $\mathbb{R}^3$ are bases of $\mathbb{R}^3$ and some are not. How do we know?

Theorem 3.1.1. *Let $B = \{v_1, \dots, v_n\}$ be any set of n vectors in $\mathbb{F}^n$. Then B is a basis of $\mathbb{F}^n$ if and only if the matrix A whose columns are $v_1, \dots, v_n$ has an inverse in $M_n(\mathbb{F})$.*

Proof. Suppose that A has an inverse of in $M_n(\mathbb{F})$. Then $AA^{-1} = I_n$, and $Aw_1 = e_1$, where w_1 is the first column of A^{-1}. It follows that e_1 is a linear combination of $v_1, \dots, v_n$. Similarly each e_i is in the linear span of $\{v_1, \dots, v_n\}$, and so $\{v_1, \dots, v_n\}$ is a spanning set of $\mathbb{F}^n$. Hence it is a basis of $\mathbb{F}^n$ by Lemma 2.3.1.

On the other hand, suppose that B is a basis of $\mathbb{F}_n$. Then e_1 is a linear combination of the columns of B, and so $e_1 = Bw_1$, for some $w_1 \in \mathbb{F}^n$. Similarly $e_i = Bw_i$, for $i = 2, \dots, n$. It follows that $AW = I_n$, where W is the matrix in $M_n(\mathbb{F})$ whose columns are $w_1, \dots, w_n$, and hence A has an inverse in $M_n(\mathbb{F})$. □

3.1.2 Moving between two bases - an example

Suppose we have another basis $\mathcal{B} = \{b_1, b_2, b_3\}$ of R^3 (besides the standard basis), where

$$b_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

You can check that $\mathcal{B}$ is linearly independent, hence is is a basis of $\mathbb{R}^3$ - for example by checking that the RREF of the 3×3 matrix $[b_1\ b_2\ b_3]$ is I_3. We write B for the matrix with columns b_1, b_2, b_3.

Question: Suppose we have some other vector in $\mathbb{R}^3$, for example $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

What are the coordinates of v with respect to $\mathcal{B}$?

Another Question: Why would we want to know this?
Partial answer - the standard basis is very useful for example for describing a rotation of $\mathbb{R}^3$ through 180° about the Z-axis. We can say exactly how this affects each of the standard basis vectors. But if we wanted to describe a rotation around a different axis, say for example one in the direction of b_1 (which is perpendicular to b_2 and b_3) maybe the standard basis is not the best for that. We will come back to this theme shortly, for now the suggestion is to just keep it in mind.

Back to the first question: if we knew how to write e_1, e_2 and e_3 as a linear combination of b_1, b_2, b_3, we could do the same for v (or any vector). To figure this out: the $\mathcal{B}$-coordinates of e_1 are the values of x, y, z in the unique solution of

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ or } B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = e_1.$$

The corresponding values are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which means they are the entries of Column 1 of B^{-1}. In the same way, the $\mathcal{B}$-coordinates of e_1 and e_3 are given by Columns 2 and 3 of B^{-1}.

We can confirm this for our example:

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \ B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

Looking at (for example) Column 2 of B^{-1} we can confirm that its entries are the $\mathcal{B}$-coordinates of e_2:

$$1b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3 = 1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2} - \frac{1}{2} \\ 1 - \frac{1}{2} + \frac{1}{2} \\ -1 + 1 \end{bmatrix} = e_2.$$

Now for the $\mathcal{B}$-coordinates of $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. We write $[v]_{\mathcal{B}}$ for the column whose entries are the $\mathcal{B}$-coordinates of v. The punchline is that we can now achieve this through a matrix-vector product.

$$\begin{aligned} v = 2e_1 + 1e_2 + 3e_3 \implies [v]_{\mathcal{B}} &= 2[e_1]_{\mathcal{B}} + 1[e_2]_{\mathcal{B}} + 3[e_3]_{\mathcal{B}} \\ &= \begin{bmatrix} [e_1]_{\mathcal{B}} & [e_2]_{\mathcal{B}} & [e_3]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

Conclusion: $v = 6b_1 + \frac{9}{2}b_2 + \frac{1}{2}b_3$.

Exercise: Confirm this conclusion by direct calculation.

More important conclusion: To find the $\mathcal{B}$-coordinates of *any* vector v in $\mathbb{R}^3$, what we need to do is multiply v on the left by the *change of basis* matrix from the standard basis to $\mathcal{B}$. This is the inverse of the matrix whose columns are the elements of $\mathcal{B}$ (written in the standard basis).

LEARNING OUTCOMES FOR THIS SECTION

1. *How to recognize when a set of n column vectors in $\mathbb{R}^n$ (or $\mathbb{F}^n$) forms a basis.*
   Think about the statement of Theorem 3.1.1 first, and try out a few of your own examples with $n = 2$ or $n = 3$, to get a sense of what it is saying. This is a good thing to do before studying the proof (which is not necessarily urgent).

2. *To recognize that elements of $\mathbb{R}^n$ (or $\mathbb{F}^n$) have different coordinates with respect to different bases*
   Think about this for some examples with $n = 2$.

3. *To use the change of basis matrix to write the coordinates of any vector in $\mathbb{F}^n$ with respect to a given basis*
   The instruction for how to do this is the "More important conclusion" above. Try it out for other vectors besides the v that was used here. Use a couple of such examples to satisfy yourself that it works, then go over the steps to think about *why* it works.

3.2 The Rank-Nullity Theorem

Let V and W be $\mathbb{F}$-vector spaces and let $\phi : V \to W$ be a linear transformation. Recall what this means:

- $\phi(u + v) = \phi(u) + \phi(v)$ for all $u, v \in V$, and $\phi(\lambda v) = \lambda\phi(v)$, for all $v \in V$ and $\lambda \in \mathbb{F}$.

Example 3.2.1. *If A is matrix in $M_m \times n(\mathbb{F})$, then left multiplication by A defines a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. For example, the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix}$ defines a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ via*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a + 3b + c \\ a - 2b + c \end{bmatrix}.$$

For example, $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \to \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ under this transformation.

Note that the images of the three standard basis vectors of $\mathbb{R}^3$ under this transfomation are respectively the columns of A.

Now suppose that $\dim V = n$ and $\dim W = m$. Once we choose bases $\mathcal{B}_V$ and $\mathcal{B}_W$ for V and W, *every* linear transformation from V to W looks like the one in Example 3.2.1 above. For example, the differential operator D, which sends every polynomial to its derivative, is a linear transformation from $\mathbb{R}[x]$ to $\mathbb{R}[x]$. But $\mathbb{R}[x]$ is an infinite-dimensional space, so we'll restrict our attention to the subspace P_3, which has dimension 4 and consists of all polynomials $a_1x^3 + a_2x^2 + a_3x + a_4$, of degree at most 3. The differential operator maps P_3 to P_2 (polynomials of degree at most 2).

Now write $\mathcal{B}_3 = \{x^3, x^2, x, 1\}$ and $\mathcal{B}_2 = \{x^2, x, 1\}$ - bases for P_3 and P_2 respectively. We take each of the four basis elements of $\mathcal{B}_3$ and look at its image in P_2 under D, considered as a vector in terms of its $\mathcal{B}_2$-coordinates. We have

$$x^3 \to 3x^2 \leftrightarrow \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}_2}, \ x^2 \to 2x \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_{\mathcal{B}_2}, \ x \to 1 \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}_2}, \ 1 \to 0 \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}_2}.$$

The $\mathcal{B}_3$-coordinates of the element $p(x) = ax^3 + bx^2 + cx + d$ are given by the column $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, and the $\mathcal{B}_2$ coordinates of the derviative of p are given by

$$a \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}_2} + b \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_{\mathcal{B}_2} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}_2} + d \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}_2} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{\mathcal{B}_3}.$$

The 3×4 matrix above is the matrix of D with respect to the bases $\mathcal{B}_3$ and $\mathcal{B}_2$. Its columns are the images under D of the elements of $\mathcal{B}_2$, written with respect to $\mathcal{B}_3$. To apply the operator to any polynomial $p(x)$, we can write it as a column vector (with respect to $\mathcal{B}_3$) and then multiply by the matrix. The result has the $\mathcal{B}_2$-coordinates of $p'(x)$.

Important Note: This matrix depends on the choice of bases! Suppose we keep the basis $\mathcal{B}_2$ of P_2, but take $\mathcal{C}_3 = \{x^3+x^2, x^2+x, x+1, 1\}$ as our basis of P_3. The matrix of the differential operator with respect to this choice has the $\mathcal{B}_2$-coordinates of the derivatives of elements of $\mathcal{C}_3$ as its columns, it is given by

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

To use this matrix to determine the derivative of (for example) $f(x) = x^3 + 4x^2 - x - 2$, first write $f(x)$ with respect to $\mathcal{C}_3$: $1(x^3 + x^2) + 3(x^2 + x) - 4(x + 1) + 2(1)$. Then

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix}_{\mathcal{C}_3} = \begin{bmatrix} 3 \\ 8 \\ -1 \end{bmatrix}_{\mathcal{B}_2}.$$

The key points of this example are:

- that every linear transformation becomes a matrix product once bases have been chosen for the domain and target spaces, and
- that the matrix involved depends on the choice of bases,

and definitely not that this is a recommended method for differentiating polynomials, especially not the second version of it!

Now suppose that $\Phi : V \to W$ is a linear transformation of finite dimensional vector spaces. There are a couple of important subspaces, of U and V respectively, associated with Φ.

Definition 3.2.2. *The* kernel *of* Φ *, denoted* $\ker \Phi$*, is the set of elements of* V *whose image is the zero vector of* W*.*

$$\ker \Phi = \{v \in V : \Phi(v) = 0_W\} \subseteq V.$$

Definition 3.2.3. *The* image *of* Φ *, denoted* image Φ*, is the subset of* W *consisting of the images of all the elements of* V*.*

$$\text{image } \phi = \{\phi(v) : v \in V\} \subseteq W.$$

Since every linear transformation can be defined in terms of matrices, the concepts of kernel and image also have a matrix version

Example 3.2.4. *For the linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ *defined as left multiplication by the matrix* $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix}$ *(as in Example 3.2.1 above), the* kernel *consists of all vectors* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *for which*

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the matrix context, this is referred to as the (right) nullspace *of* A*.*

We can find it by row reduction; in this example it consists of all vectors of the form $t \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$ *where* $t \in \mathbb{R}$ *- a vector subspace of dimension 1 of* $\mathbb{R}^3$*.*

The image *of this linear transformation is the subspace of* $\mathbb{R}^2$ *consisting of all products of* A *with vectors in* $\mathbb{R}^3$ *- this is the linear span of the three columns of* A*. In the matrix context, it is called the* column space *of* A*. In this example, it is all of* $\mathbb{R}^2$*, since the first two columns of* A *(for example) span* $\mathbb{R}^2$*.*

We can note that in this example, the kernel (nullspace) and image (columnspace) have dimension 1 and 2, and 1+2=3, and 3 is the dimension of the domain $\mathbb{R}^3$. *This is not a coincidence, but a case of the* Rank-Nullity Theorem.

If $\phi : V \to W$ is a linear transformation, the kernel and image of Φ are *subspaces* of V and W respectively. To see this:

- for $\ker \phi$: Suppose that $u, v \in \ker \phi$. Then

  $$\phi(u+v) = \phi(u) + \phi(v) = 0_W + 0_W = 0_w,$$

  so $\ker \phi$ is closed under addition in V.
  If $u \in \ker \phi$ and $\lambda \in \mathbb{F}$, then

  $$\phi(\lambda u) = \lambda\phi(u) = \lambda 0_W = 0_W,$$

  so $\lambda u \in \ker \phi$ and $\ker \phi$ is closed under multplication by scalars.

- for image ϕ: Suppose that $w, z \in$ image ϕ. Then $w = \phi(u)$ and $z = \phi(v)$ for some u and v in V, and

  $$w + z = \phi(u) + \phi(v) = \phi(u+v),$$

  so $w + z \in$ image ϕ and image ϕ is closed under addition in W.
  If $w \in \ker \phi$ and $\lambda \in \mathbb{F}$, then $w = \phi(u)$ for $u \in V$, and

  $$\lambda w = \lambda\phi(u) = \phi(\lambda u),$$

  so $\lambda w \in$ image ϕ and image ϕ is closed under multplication by scalars.

Now we come to the Rank-Nullity Theorem, which relates the dimensions of the kernel, image and domain of a linear transformation. Equivalently, it relates the dimensions of the nullspace and column space of a matrix to the number of columns. The dimension of the image of a linear trasformation is called its *rank*, and the dimension of the kernel is called the *nullity*.

Theorem 3.2.5. *Rank-Nullity Theorem Let $\phi : V \to W$ be a linear transformation, where V and W are finite-dimensional vector spaces over a field $\mathbb{F}$. Then*

$$\dim(\ker \phi) + \mathrm{rank}\phi = \dim V.$$

Proof. Write n for $\dim V$ and k for $\dim(\ker \phi)$. Let $\{v_1, \ldots v_k\}$ be a basis of $\ker \phi$. This may be extended to a basis $\mathcal{B} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. Since $\mathcal{B}$ spans V, every element of image phi has the form

$$\begin{aligned} \phi(a_1v_1 + \cdots + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n) &= \phi(a_{k+1}v_{k+1} + \cdots + a_nv_n) \\ &= a_{k+1}\phi(v_{k+1}) + \cdots + a_n\phi(v_n), \end{aligned}$$

for some scalars $a_{k+1}, \ldots, a_n$ (and $a_1, \ldots, a_k$). It follows that the set $\mathcal{B}' = \{\phi(v_k), \ldots, \phi(v_n)\} \subseteq W$ is a spanning set of image ϕ. We now show that $\mathcal{B}'$ is a basis of image ϕ, by showing that it is linearly independent. Suppose not, and suppose that

$$c_{k+1}\phi(v_{k+1}) + \cdots + c_n\phi(v_n) = 0_W$$

for some scalars $c_{k+1}, \ldots, c_n$. Then

$$\phi(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0_W \Longrightarrow c_{k+1}v_{k+1} + \cdots + c_nv_n \in \ker \phi.$$

But this means that $c_{k+1}v_{k+1} + \cdots + c_nv_n$ is a linear combination of $v_1, \ldots, v_k$, contrary to the linear independence of $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$.

We conclude that $\mathcal{B}'$ is a basis of image ϕ, which means that the image of ϕ has dimension $n - k$ (this is the rank of ϕ), and so

$$\dim(\ker \phi) + \mathrm{rank}\phi = k + n - k = n = \dim V,$$

as required. □

Theorem 3.2.6. *Rank-Nullity Theorem, matrix version Let A be any $m \times n$ matrix, with entries in a field $\mathbb{F}$. Then n is the sum of the dimension of the right nullspace of A and the dimension of the column space of A.*

The dimension of the columns space of a matrix A is called the *column rank* of A.

LEARNING OUTCOMES FOR THIS SECTION

1. *To recall the defintion of a linear transformation*
   as a function between vector spaces that respects the addition and scalar multiplication operations.
2. *To note that left multiplication by any $m \times n$ matrix is a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$*,
   and that the columns of the matrix are the images of the standard basis vectors of $\mathbb{F}^n$
3. *That every linear transformation can be represented as left multiplication by a matrix*,
   after choosing bases for the domain and target spaces. For relatively small and manageable examples, you should be able to write down the matrix that does this, and realize that it depends on the choice of basis (we will come back to this point).
4. *To recognize the terms kernel, image, nullspace, nullity, rank and column space*.
5. *To be able to state and interpret the Rank-Nullity Theorem, in its versions for matrices and for linear transformations*
   The proof is important too, but understanding the statement is more important. One way to think of it informally is that if we apply a linear transformation to a space of dimension n, the image need not have the full dimension n, because some of the elements might be mapped to zero, and so not be "recoverable" in the image (these are the elements of the kernel). But the full dimension n has to be accounted for by the combination of the kernel or the image - their dimensions must add up to n.

3.3 Similarity

In this section we will consider the algebraic relationship between two square matrices that represent the same linear transformation, from a vector space to itself, with respect to different bases.

Example 3.3.1. *Let $T : R^3 \to \mathbb{R}^3$ be the linear transformation defined by $v \to Av$, where*

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix}$$

Let $\mathcal{B}$ be the (ordered) basis of $\mathbb{R}^3$ with elements $b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

What is the matrix A' of T with respect to $\mathcal{B}$?

The columns of A' have the $\mathcal{B}$-coordinates of $T(b_1), T(b_2)$ and $T(b_3)$.

$$T(b_1) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = 2b_1 \Longrightarrow [T(b_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -3b_2 \Longrightarrow [T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$T(b_3) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 14 \end{bmatrix} = 7b_3 \Longrightarrow [T(b_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

We conclude that the matrix of T with respect to $\mathcal{B}'$ is

$$A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

This means: for any $v \in \mathbb{R}^3$,

$$[T(v)]_{\mathcal{B}} = A'[v]_{\mathcal{B}}.$$

The relation of similarity. Staying with this example for now, we consider the relationship between A and A′ from another viewpoint. Let P be the matrix with the basis vectors from $\mathcal{B}$ as columns. From Section 3.1, we know that P^{-1} is the change of basis matrix from the standard basis to $\mathcal{B}$. This means that for any element v of $\mathbb{R}^3$, its $\mathcal{B}$-coordinate are given by the matrix-vector product

$$[v]_{\mathcal{B}} = P^{-1}v.$$

Equivalently, if we start with the $\mathcal{B}$-coordinates, then the standard coordinates of v are given by

$$v = P[v]_{\mathcal{B}}.$$

So P itself is the change of basis matrix from $\mathcal{B}$ to the standard basis. Suppose we only knew about A (and had not already calculated A′). We have a basis $\mathcal{B}$ whose columns form the matrix P. To figure out the matrix of T with respect to $\mathcal{B}$:

1. Start with an element of $\mathbb{R}^3$, written in its $\mathcal{B}$-coordinates: $[v]_{\mathcal{B}}$
2. Convert the vector to its standard coordinates (so that we can apply T by multiplying by A): this means taking the product $P[v]_{\mathcal{B}}$
3. Now apply T: this means taking the product $AP[v]_{\mathcal{B}}$. This vector has the standard coordinates of $T(v)$.
4. To convert this to $\mathcal{B}$-coordinates, apply the change of basis matrix from standard to $\mathcal{B}$, which is P^{-1}: this means taking the product $P^{-1}AP[v]_{\mathcal{B}}$. This vector has the $\mathcal{B}$-coordinates of $T(v)$.
5. Conclusion: For any element v of $\mathbb{R}^3$, the $\mathcal{B}$-coordinates of $T(v)$ are given by

$$(P^{-1}AP)[v]_{\mathcal{B}}.$$

This conclusion is saying that the matrix of T with respect to $\mathcal{B}$ is $P^{-1}AP$, where A is the matrix of T with respect to the standard basis, and P is the matrix with the (standard) elements of $\mathcal{B}$ as columns.

Definition 3.3.2. *Let $\mathbb{F}$ be a field. Two matrices A and B in $M_n(\mathbb{F})$ are* similar *if there exists an invertible matrix $P \in M_n(\mathbb{F})$ for which $B = P^{-1}AP$.*

Notes

1. Two distinct matrices in $M_n(\mathbb{F})$ are similar if and only if they represent the same linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^n$, with respect to different bases.
2. As the examples A and A′ above show, it is not generally easy to tell by glancing at a pair of square matrices whether they are similar or not, but there is one feature that is easy to check. The *trace* of a square matrix is the sum of the entries on the main diagonal, from top left to bottom right. If two matrices are similar, they have the same trace.
3. Similar matrices also have some other features in common, including having the same determinant. But we have not discussed determinants yet (coming soon).

Item 2. above is a consequence of the following lemma.

Lemma 3.3.3. *Let $A, B \in M_n(\mathbb{F})$. Then* $\text{trace}(AB) = \text{trace}(BA)$.

Consequence: For any square matrix A and any invertible matrix P, both in $M_n(\mathbb{F})$, $\text{trace}(P^{-1}AP) = \text{trace}(AP)P^{-1} = \text{trace}\, A$, so similar matrices always have the same trace.

Proof. (of the Lemma). We calculate the trace of AB in terms of the entries of A and B.

$$\begin{aligned} \text{trace}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} (sum_{k=1}^{n} A_{ik} B_{ki}) . \end{aligned}$$

This is the sum over all positions (i, k) of a $n \times n$ matrix of the expressions

(entry in (i, k)-position of A)$\times$ (entry in (k, i)-position of B).

This sum does not change if the roles of A and B are switched, so AB and BA have the same trace. □

In Example 3.3.1, we found that the 3×3 matrix A is similar to the diagonal matrix $A' = \text{diag}(2. -3, 7)$. We say that A is *diagonalizable*, which means that it is similar to a diagonal matrix. It $T : \mathbb{R}^3 \to \mathbb{R}^3$ is left multiplication by A, then A' is the matrix of T with respect to the basis $\mathcal{B} = (b_1, b_2, b_3)$, and the basis elements b_1, b_2, b_3 are the columns of the matrix P for which $P^{-1}AP = A'$.
Two (equivalent) observations about this setup:

1. From the diagonal form of A' we have $T(b_1) = 2b_1$, $T(b_2) = -3b_2$ and $T(b_3) = 7(b_3)$. This means that each of the basis elements b_1, b_2, b_3 is mapped by T to a scalar multiple of itself - each of them is an *eigenvector* of T.

2. We can rearrange the version $P^{-1}AP = A'$ to $AP = PA'$. Bearing in mind that $P = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$ and that $A' = \text{diag}(2, -3, 7)$, this is saying that

$$A \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \implies \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ 2b_1 & -3b_2 & 7b_3 \\ | & | & | \end{bmatrix}$$

   This means that $Ab_1 = 2b_1$, $Ab_2 = -3b_2$ and $Ab_3 = 7b_3$, so that $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$ consisting of *eigenvectors* of A.

Definition 3.3.4. *Let* $T : V \to V$ *be a linear transformation from a vector space* V *to itself. An* eigenvector *of* T *is a non-zero element* v *of* V *for which* $T(v) = \lambda v$ *for some scalar* λ *(called the* eigenvalue *of* T *to which* v *corresponds).*

In this situation, T can be represented by a diagonal matrix if and only if V has a basis consisting of eigenvectors of T.

Definition 3.3.5. *(Matrix Version). Let* $A \in M_n(\mathbb{F})$*. An* eigenvector *of* A *is a non-zero vector* $v \in \mathbb{F}^n$ *for which* $Av = \lambda v$ *for a scalar* λ *(called the* eigenvalue *of* A *to which* v *corresponds).*

The matrix A is diagonalizable (similar to a diagonal matrix) if and only if there is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

LEARNING OUTCOMES FOR SECTION 3.3

1. *To be able to explain the meaning of the matrix of a linear transformation with respect to a particular basis*

2. *To be able to describe the relation of similarity for matrices*
   and explain its meaning in terms of linear transformations.

3. *To know what it means for a matrix (or linear transformation) to be diagonalizable.*

3.4 Eigenvectors

Definition 3.4.1. *Let* $T : V \to V$ *be a linear transformation, where* V *is a finite dimensional vector space. A non-zero element* v *of* V *is a* eigenvector *of* T *if* $T(v)$ *is a scalar multiple of* v*.*

If v is an eigenvector of T, then the 1-dimensional subspace of V spanned by v, which consists of all scalar multiples of v, is mapped to itself by T. It is said to be a T-*invariant* line.

Definition 3.4.2. *If* v *is an eigenvector of* T*, then* $T(v) = \lambda v$ *for some scalar* λ*, and* λ *is called the* eigenvalue *of* T *to which* v *corresponds.*

Here is the matrix version.

Definition 3.4.3. *Let* $A \in M_n(\mathbb{F})$*. A vector* $v \in \mathbb{F}^n$ *is an* eigenvector *of* A *if* $Av = \lambda v$ *for some scalar* $\lambda \in \mathbb{F}$*, called the* eigenvalue *of* A *to which* v *corresponds.*

Given a matrix A and a vector v, it is quite a straightforward task to determine whether v is an eigenvector of A, and to determine the corresponding eigenvalue if so - just calculate the matrix-vector product Av and see if it is a scalar multiple of v. In fact, given a vector v, we can construct a matrix that has v as an eigenvector, with our favourite scalar as an eigenvalue.

Example 3.4.4. *Find a matrix* $A \in M_3(\mathbb{R})$ *that has* $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *as an eigenvector, corresponding to the eigenvalue 28.*

To do this, write u_1, u_2, u_3 as the three rows of A. What we need is that $u_1 v = 28(1) = 28$, $u_2 v = 28(2) = 56$, $u_3 v = 28(3) = 84$. The easy way to arrange this is to choose $u_1 = (28\ \ 0\ \ 0)$, $u_2 = (0\ \ 28\ \ 0)$, $u_3 = (0\ \ 0\ \ 28)$, so that $A = 28I_3$. This answer is correct but we can find others, and the conditions on u_1, u_2, u_3 are independent. For example we can choose

$$u_1 = (3\ \ 2\ \ 3),\ \ u_2 = (0\ \ -2\ \ 20),\ \ u_3 = (5\ \ 2\ \ 25)$$

to get $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & -2 & 20 \\ 5 & 2 & 25 \end{bmatrix}$, and it is easily confirmed that $Av = 28v$.

Exercise: Show that the set of matrices in $M_3(\mathbb{R})$ that satisfy $Mv = 28v$ is a *subspace* of $M_3(\mathbb{R})$.

Example 3.4.5. Show that $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} -2 & 9 \\ 8 & 4 \end{bmatrix}$ and find the corresponding eigenvalue.

$$\begin{bmatrix} -2 & 9 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 10 \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

The corresponding eigenvalue is 10.

Diagonal matrices.

A harder problem is to find the eigenvectors of a matrix or linear trasformation, given only the matrix or linear transformation itself. For example, suppose that

$$B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}.$$

Finding an eigenvector of B means finding solutions for x, y, z and λ, to the following system of equations, where the values of x, y, z are not all zero.

$$\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If λ is regarded as a variable, this is not a system of linear equations. Where to begin?

It turns out that the key to making progress is to find the *eigenvalues* first, even if it's the eigenvectors that we want. To see why, we show that the number of distinct eigenvalues of a $n \times n$ matrix cannot exceed n.

Theorem 3.4.6. *Let $A \in M_n(\mathbb{F})$ and let $v_1, \dots, v_k$ be eigenvectors of A in $\mathbb{F}^n$, corresponding to* distinct *eigenvalues $\lambda_1, \dots, \lambda_k$ of A. Then $\{v_1, \dots, v_k\}$ is a linearly independent subset of $\mathbb{F}^n$.*

Idea of Proof: First suppose that $k = 2$, and suppose that $a_1 v_1 + a_2 v_2 = 0$, for scalars a and b in $\mathbb{F}$. We need to show that $a_1 = a_2 = 0$. Multiplying the expression $a_1 v_1 + a_2 v_2$ on the left by A, we have

$$a_1 A v_1 + a_1 A v_2 = 0 \Longrightarrow a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0.$$

Multiplying the same expression by the scalar λ_1 gives

$$a_1 \lambda_1 v_1 + a_2 \lambda_1 v_2 = 0.$$

Subtracting one of these expressions from the other gives

$$a_2(\lambda_1 - \lambda_2) v_2 = 0.$$

Now v_2 is not the zero vector because it is an eigenvector of A, and $\lambda_1 - \lambda_2$ is not zero, because λ_1 and λ_2 are distinct eigenvalues. So it must be that $a_2 = 0$. Since $a_1 v_1 + a_2 v_2 = 0$, it follows that $a_1 = 0$ also, since v_1 is not the zero vector (begin an eigenvector of A). We conclude that the zero vector can be written as a linear combination of v_1 and v_2 only if both coefficients are zero, which means that $\{v_1, v_2\}$ is a linearly independent set.

The proof in the general situation uses exactly this idea.

Proof. If $\{v_1, \dots, v_k\}$ is linearly dependent, then there are expressions for the zero vector as a linear combination of $v_1, \dots, v_k$ in which the coefficients are *not* all zero. Let d be the least number of non-zero coefficients in any such expression, and (after reordering the v_i and λ_i if necessary), suppose that

$$a_1 v_1 + \cdots + a_d v_d = 0,$$

with $d \geqslant 2$ and each a_i is a non-zero element of $\mathbb{F}$. Multiplying this equation respectively by A (on the left) and by λ_1 gives

$$\begin{aligned} a_1\lambda_1 v_1 + a_2\lambda_2 v_2 + \cdots + a_d\lambda_d v_d &= 0 \\ a_1\lambda_1 v_1 + a_2\lambda_1 v_2 + \cdots + a_d\lambda_1 v_d &= 0 \end{aligned}$$

Subtracting the second equation from the first gives

$$a_2(\lambda_2 - \lambda_1)v_1 + a_3(\lambda_3 - \lambda_1)v_2 + \cdots + a_d(\lambda_d - \lambda_1)v_d = 0.$$

None of the coefficients in this linear combination of $v_2, \dots, v_d$ are zero, since the a_i are all non-zero and the λ_i are all distinct. So this is a non-trivial expression for the zero vector as a linear combination of $v_1, \dots, v_k$ with fewer than d non-zero coefficients, which contradicts the choice of d. We conclude that $\{v_1, \dots, v_k\}$ is a linearly independent subset of $\mathbb{F}^n$. $\square$

The following consequence of Theorem 3.4.6 suggests that we may have some chance of being able to find the eigenvalues of a $n \times n$ matrix, or at least that there are not too many of them.

Corollary 3.4.7. *Let $A \in M_n(\mathbb{F})$. Then A has at most n distinct eigenvalues in $\mathbb{F}$.*

Proof. If A has k distinct eigenvalues, with corresponding eigenvectors $v_1, \dots, v_k$ in $\mathbb{F}^n$, then k cannot exceed the dimension of $\mathbb{F}^n$, since $\{v_1, \dots, v_k\}$ is a linearly indepedent set in $\mathbb{F}^n$. Hence $k \leqslant n$. $\square$

The following consequence is also important and useful.

Corollary 3.4.8. *Let $A \in M_n(\mathbb{F})$ and suppose that A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ in $\mathbb{F}$. Then A is diagonalizable, and A is similar to the matrix* $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

Proof. Let $v_1, \dots, v_n$ be eigenvectors of A in $\mathbb{F}^m$, corresponding to $\lambda_1, \dots, \lambda_n$ respectively. Then $(v_1, \dots, v_n)$ is an (ordered) basis of $\mathbb{F}^n$, by Theorem 3.4.6. If P is the matrix with columns $v_1, \dots, v_n$, then $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. $\square$

If a $n \times n$ matrix has fewer than n distinct eigenvalues, then it may or may not be diagonalizable. The two examples below indicate two ways in which a matrix in $M_n(\mathbb{F})$ could fail to be diagonalizable in $M_n(\mathbb{F})$.

Example 3.4.9. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $M_2(\mathbb{R})$.

Suppose that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of A. Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{array}{rcl} x + y & = & \lambda x \\ y & = & \lambda y \end{array}$$

The second equation says that $y = 0$ or $\lambda = 1$. If $y = 0$, then the first equation says $x = \lambda x$. Since x and y cannot both be 0 in an eigenvector, it follows that $\lambda = 1$ anyway. Thus $\lambda = 1$ is the *only* possible eigenvalue of A. The non-zero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 1$ if and only if $x + y = x$ and $y = y$. The first equation says $y = 0$, and x may have any value. The eigenvectors of A are all vectors of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$, where $x \neq 0$ in $\mathbb{R}$, i.e. all scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. These comprise only a 1-dimensional subspace of $\mathbb{R}^2$, os $\mathbb{R}^2$ does not have a basis consisting of eigenvectors of A, and A is not diagonalizable.

The point of the following example is to show that if A is a matrix in $M_n(\mathbb{F})$, the eigenvalues of A may not be in $\mathbb{F}$ but in a bigger field.

Example 3.4.10. Let $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in $M_2(\mathbb{R})$.

Note that B is the matrix of a counter-clockwise rotation through $\frac{pi}{2}$ about the origin in $\mathbb{R}^2$. From that geometric intepretation we can see that B has no eigenvector in $\mathbb{R}^2$, since no line in $\mathbb{R}^2$ is preserved by this rotation. We can also see this algebraically.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{array}{rcl} y & = & \lambda x \\ -x & = & \lambda y \end{array}.$$

Looking at both of these equations, we have $y = \lambda x = \lambda(-\lambda y) \longrightarrow y = -\lambda^2 y$.

If $y = 0$, then $x = 0$ which does not give an eigenvector. If $y \neq 0$, then $y = -\lambda^2 y$ means $\lambda^2 = -1$, which is not satisfied by any real number λ. This means that B has no real eigenvalue and no eigenvector in $\mathbb{R}^2$. However, if we allow complex values for λ, then $\lambda = i$ and $\lambda = -i$ satisfy $\lambda^2 = -1$. To find corresponding eigenvectors:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = i \begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{array}{rcl} y & = & ix \\ -x & = & iy \end{array}$$

So $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue i.

For the eigenvalue $-i$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -i \begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{array}{rcl} y & = & -ix \\ -x & = & -iy \end{array}$$

So $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue i.

We conclude that B is not diagonalizable in $M_2(\mathbb{R})$ but that it is similar in $M_2(\mathbb{C})$ to the diagonal matrix $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Learning Outcomes for Section 3.4

1. To define an eigenvector of a linear transformation or of a square matrix.
2. To know that eigenvectors corresponding to different eigenvalues are linearly independent.
3. And that this means a $n \times n$ matrix can have at most n distinct eigenvalues
4. and that it is diagonalizable if it does have n distinct eigenvalues.

3.5 The Characteristic Polynomial

In this section we will discuss how to determine the eigenvalues of a given matrix. In practice, we cannot always precisely determine them, but we can write down a polynomial whose coefficients depend on the entries of the matrix, and whose roots are the eigenvalues.

Example 3.5.1. *Find a matrix* P *with* $P^{-1}AP$ *diagonal, where* $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

To answer this, we need to find two linearly independent eigenvectors of A. These are non-zero solutions of

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{matrix} 2x+2y & = & \lambda x \\ x+3y & = & \lambda y \end{matrix} \Longrightarrow \begin{matrix} 0 & = & (\lambda-2)x-2y \\ 0 & = & -x+(\lambda-3)y \end{matrix} \Longrightarrow \begin{bmatrix} \lambda-2 & -2 \\ -1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So we are looking for non-zero solutions $\begin{bmatrix} x \\ y \end{bmatrix}$ of the system

$$\begin{bmatrix} \lambda-2 & -2 \\ -1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These can occur only if the coefficient matrix is non-invertible. If it is invertible, the only solution is $x = y = 0$.

A 2×2 matrix is non-invertible if and only if its determinant is 0. The *determinant* of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

$$\det\begin{bmatrix} \lambda-2 & -2 \\ -1 & \lambda-3 \end{bmatrix} = (\lambda-2)(\lambda-3)-(-2)(-1) = \lambda^2-5\lambda+4.$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1).$$

The *eigenvalues* of A are the solutions of the *characteristic equation* $\det(\lambda I - A) = 0$, 1 and 4. The *eigenspace* of A corresponding to $\lambda = 1$ is the set of all solutions of the system

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{bmatrix} 1-2 & -2 \\ -1 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the nullspace of the matrix $1I - A = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$, which is

$$\left\{\begin{bmatrix} -2t \\ t \end{bmatrix}, t \in \mathbb{R}\right\}.$$

An eigenvector of A for $\lambda = 1$ is any non-zero element of this space, for example $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The *eigenspace* of A corresponding to $\lambda = 4$ is the nullspace of the matrix $4I - A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$, which is

$$\left\{\begin{bmatrix} t \\ t \end{bmatrix}, t \in \mathbb{R}\right\}.$$

An eigenvector of A for $\lambda = 4$ is any non-zero element of this space, for example $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Conclusion: If P is a matrix whose columns are eigenvectors of A corresponding respectively to the eigenvalues 1 and 4, for example $P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Multiplying each column of this P by a non-zero scalar gives alternative choices of P, with the same diagonal matrix $P^{-1}AP$. Switching the two columns of P would give a matrix Q with $Q^{-1}AQ = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

3.5.1 The Determinant (a digression)

For any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2.$$

From this matrix equation we have the following observations:

- If $ad - bc = 0$, then A is not invertible, because the columns $\begin{bmatrix} d \\ -c \end{bmatrix}$ and $\begin{bmatrix} -b \\ a \end{bmatrix}$ are in its nullspace (and these are both zero only if A is the zero matrix).
- If $ad - bc \neq 0$, then the equation shows that $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The matrix A has an inverse if and only if $ad - bc \neq 0$. This means that the number $ad - bc$ tells us whether or not the columns of A form a basis of $\mathbb{F}^2$ (or $\mathbb{R}^2$).

The equation above also prompts the following definitions

- The number (or field element) $ad - bc$ is the *determinant* of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denoted by $\det(A)$ or sometimes $|A|$.
- The matrix $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the *adjugate* (sometimes called the *adjoint*) of A, denoted by $\mathrm{adj}(A)$.

The version of the above equation for a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is the following:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} ei - fh & -bi + ch & bf - ce \\ -di + fg & ai - cg & -af + cd \\ dh - eg & -ah + bg & ae - bd \end{bmatrix} = (aei - afh - bdi + bfg + cdh - ceg)I_3.$$

This equation can be checked directly. The expression $(aei - afh - bdi + bfg + cdh - ceg)$ is the *determinant* of A, and the adjugate of A is the matrix on the right. Its entries are the determinants of the nine 2×2 submatrices of A (some with a sign change). To see why this definition of the 3×3 determinant is consistent with the 2×2 version, we can write it as follows:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix} = (\underbrace{aei - afh - bdi + bfg + cdh - ceg}_{\det(A)})I_3.$$

Definition 3.5.2. *The* minor $M_{i,j}$ *of the entry in the* (i, j) *position of a* 3×3 *matrix* A *is the determinant of the* 2×2 *matrix that remains when Row* i *and Column* j *are deleted from* A.

Definition 3.5.3. *The* cofactor $C_{i,j}$ *of the entry in the* (i, j) *position of a* 3×3 *matrix* A *is either equal to* $M_{i,j}$ *or to* $-M_{i,j}$, *according to the following pattern of signs:* $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Definition 3.5.4. *The* adjugate *of the* 3×3 *matrix* A *is the matrix that has* $C_{j,i}$ *in the* (i, j)*-position. It is the* transpose *of the matrix of cofactors of* A.

By looking at any of the three entries on the main diagonal of the product $A \times \text{adj}(A)$, we can give the following description of the determinant of a 3×3 matrix.

Definition 3.5.5. *The determinant of a* 3×3 *matrix is* A *can be found by choosing any row or column of* A, *multplying each entry of that row or column by its own cofactor, and adding the results.*

NOTES

1. Each of the definitions above applies to $n \times n$ matrices in general, and gives us a way to recursively define a $n \times n$ determinant, in terms of $(n-1) \times (n-1)$ determinants.
2. The *cofactor expansion* method, described in Definition 3.5.5 above, is not generally the most efficient way to compute a determinant (it is ok in the 3×3 case). But it can be taken as the *definition* of a determinant.
3. In some special cases, the determinant is easier to compute. If A is upper or lower triangular, then $\det(A)$ is the product of the entries on the main diagonal of A. If A has a square $k \times k$ block A_1 in the upper left, a square $(n-k) \times (n-k)$ block in the lower right, and only zeros in the lower left $(n-k) \times k$ region, then $\det(A) = \det(A_1)\det(A_2)$.
4. For a pair of $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$. This is the *multplicative property* of the determinant, or the Cauchy-Binet formula. It is not obvious at all.

We note the following consequence of item 4 above.

Theorem 3.5.6. *If* A *and* B *are similar matrices in* $M_n(\mathbb{F})$, *then they have the same determinant and the same characteristic polynomial.*

Proof. Since B is similar to A, $B = P^{-1}AP$ for some invertible matrix P. Then $\det(B) = \det(P^{-1}AP) = \det(APP^{-1}) = \det(A)$. The characteristic polynomial of B is $\det(\lambda I_n - P^{-1}AP) = \det(P^{1}(\lambda I_n - A)P) = \det(\lambda I_n - A)$. □

3.5.2 Algebraic and Geometric Multiplicity

Example 3.5.7. Using cofactor expansion by the first column, we find that the characteristic polynomial of $B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$ is

$$\det(\lambda I_3 - B) = \det \begin{bmatrix} \lambda - 5 & -6 & -2 \\ 0 & \lambda + 1 & 8 \\ -1 & 0 & \lambda + 2 \end{bmatrix}$$

$$\begin{aligned} &= (\lambda - 5)\left((\lambda+1)(\lambda+2) - 0(8)\right) + (-1)\left((-6)(8) - (\lambda+1)(-2)\right) \\ &= (\lambda-5)(\lambda^2 + 3\lambda + 2) - (2\lambda - 46) \\ &= \lambda^3 - 2\lambda^2 - 15\lambda + 36 \\ &= (\lambda - 3)(\lambda^2 + \lambda - 12) \\ &= (\lambda-3)(\lambda+4)(\lambda-3) \\ &= (\lambda-3)^2(\lambda+4) \end{aligned}$$

The eigenvalues of B are 3 (occurring twice as a root of the characteristic polynomial), and -4 (occurring once). We say that 3 has *algebraic multplicity* 2 and -4 has *algebraic multplicity* 1 as an eigenvalue of B. The *geometric multplicity* of each eigenvalue is the dimension of its corresponding eigenspace.

The eigenspace of B corresponding to $\lambda = 3$ is the nullspace of the matrix

$$3I_3 - B = \begin{bmatrix} -2 & -6 & -2 \\ 0 & 4 & 8 \\ -1 & 0 & 5 \end{bmatrix}$$