Lecture 7: Linear Transformations

Linear transformations are the primary functions between vector spaces that are of interest in linear algebra. For now we will stick to linear transformations between spaces of real column vectors.

Definition 2

Let m and n be positive integers. A linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies

- $T(u + v) = T(u) + T(v)$, and
- $T(\lambda v) = \lambda T(v)$,

for all u and v in $\mathbb{R}^n$, and all scalars $\lambda \in \mathbb{R}$.

The Matrix of a Linear Transformation

Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation. Then we can calculate the image under T of any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, if we know the images under T of the *standard basis vectors* $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. From the definition, we have

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = aT \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bT \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + cT \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where A is the 2×3 matrix that has the images of the three standard basis vectors as its three columns.

Matrix multplication is composition

Suppose that the images of the three standard basis vectors under T are respectively $\binom{1}{2}$, $\binom{1}{4}$ and $\binom{-2}{3}$. Then the matrix A of T is

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 4 & 3 \end{pmatrix}.$$

For any vector $v \in \mathbb{R}^3$, its image under T is the matrix-vector product Av.
Now suppose that $S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose matrix is $B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. This means that the images under S of $\binom{1}{0}$ and $\binom{0}{1}$ are respectively the two columns of S. Now the composition $S \circ T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, so it is represented by a matrix.

$$S \circ T(v) = S(Tv) = S(Av) = B(Av) = (BA)v.$$

The $n \times n$ identity matrix For a positive integer n, the $n \times n$ identity matrix, denoted I_n, is the $n \times n$ matrix whose entries in the $(1,1),(2,2),\dots,(n,n)$ positions (the positions on the main diagonal) are all 1, and whose entries in all other positions (all off-diagonal positions) are 0. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The special property that I_n has is that is an *identity element* or *neutral element* for matrix multplication. Multiplying another matrix by it has no effect. This means

- If A is any matrix with n rows, then $I_nA = A$, and
- If B is any matrix with n columns, then $BI_n = B$.
- In particular, if C is a $n \times n$ matrix, then $CI_n = I_nC = C$.

The Inverse of a Matrix

Let A be a square matrix of size $n \times n$. If there exists a $n \times n$ matrix B for which $AB = I_n$ and $BA = I_n$, then A and B are called inverses (or multiplicative inverses) of each other. If it does not already have another name, the inverse of A is denoted A^{-1}.

Example In $M_2(\mathbb{Q})$, the matrices $\begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ -\frac{5}{2} & -\frac{3}{2} \end{pmatrix}$ are inverses of each other.

Not every square matrix has an inverse. For example the 2×2 matrix $\begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix}$ does not.

Exercise Prove that a square matrix can only have one inverse.

The transpose of a matrix

Definition 3

The transpose of the $m \times n$ matrix A, which is denoted A^T, is defined to be the $n \times m$ matrix which has the entries of Row 1 of A in its first column, the entries of Row 2 of A in its second column, and so on.

Example If $A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & 4 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 2 \\ -2 & 0 \\ -3 & 4 \end{pmatrix}$.

For all relevant i and j, the (i, j) entry of A^T is the (j, i) entry of A. If A is $m \times n$, then the products AA^T and $A^T A$ always exist, and they are square matrices of size $m \times m$ and $n \times n$ respectively. Moreover, they are *symmetric*. A square matrix is symmetric if it is equal to its own transpose.

The transpose of a matrix product

Lemma 4

Let A and B be matrices for which the product AB is defined. Then $(AB)^T = B^T A^T$.

The lemma is saying that the transpose of the product A is the product of the transposes of A and B, but in the opposite order.

Proof.

Suppose that the sizes of A and B are $m \times p$ and $p \times n$ respectively. Choose an arbitrary position (i, j) in $(AB)^T$. The entry in this position is

$$\begin{aligned} (AB)^T_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{p} A_{jk} B_{ki} \\ &= \sum_{k=1}^{p} B^T_{ik} A^T_{kj} \end{aligned}$$

Elementary Row Operations and Matrix Algebra

Elementary row operations may themselves be interpreted as matrix multiplication exercises.

- We write I_m for the $m \times m$ identity matrix
- We write $E_{i,j}$ for the matrix that has 1 in the (i,j)-position and zeros everywhere else.

Theorem 5

Let A be a $m \times m$ matrix. Then elementary row operations on A amount to multiplying A on the left by $m \times m$ matrices, as follows:

1. *Mutiplying Row i by the non-zero scalar α is equivalent to multiplying A on the left by the matrix $I_m + (\alpha - 1)E_{i,i}$.*
2. *Switching Rows i and k amounts to multiplying A on the left by the matrix $I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,i}$.*
3. *Adding $\alpha\times$ Row i to Row k amounts to multiplying A on the left by the matrix $I_m + \alpha E_{k,i}$.*

Elementary Row Operations as Matrix Multiplication

Matrices of the three types described in Theorem 5 are sometimes referred to as *elementary matrices*. They are always invertible, and their inverses are also elementary matrices. The statement that every matrix can be reduced to RREF through a sequence of EROs is equivalent to saying that for every matrix A with m rows, there exists a $m \times m$ matrix B, which is a product of elementary matrices, with the property that BA is in RREF.

Exercise 6

Write down the inverse of an elementary matrix of each of the three types, and show that it is also an elementary matrix.
(Hint: Think about how to reverse an elementary row operation, with another elementary row operation).

Exercise 7

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Using Gauss-Jordan elimination to calculate matrix inverses

Suppose that $A \in M_n(\mathbb{F})$, for some field $\mathbb{F}$. If A is invertible, let $v_1, v_2, \dots, v_n$ be the columns of its inverse. Then

$$AA^{-1} = A \begin{bmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{bmatrix} = A \begin{bmatrix} | & | & \dots & | \\ Av_1 & Av_2 & \dots & Av_n \\ | & | & \dots & | \end{bmatrix} = I_n.$$

For each i, Av_i is the ith column of the identity matrix, which has 1 in position i and zeros elsewhere. This means that v_i is the solution of the linear system $Av_i = e_i$, where e_i is column i of the identity matrix, and the variables are the unknown entries of v_i.
We need to this for each column, but we can combine this into a single process by writing $e_1, e_2, \dots, e_n$ as n distinct columns in the "right hand side" of a $n \times 2n$ augmented matrix.

Example of inverse calculation

Example Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

To calculate A^{-1}, We apply Gauss-Jordan elimination to the 3×6 matrix below

$$A' = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

We conclude

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}.$$