MA283 Linear Algebra

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Dr Rachel Quinlan

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This module is an introduction to the theory, methods, practices and applications of linear algebra.

Linear algebra includes the study of matrices, and anything that can be represented or encoded with matrices and matrix calculations. Matrices are extremely useful in mathematics and computation, and linear algebra is a meeting point of many diverse areas.
Linear algebra is not just about matrices (really it is about vector spaces) but matrix algebra is a concrete access point to it. We will start there.

A few practical problems tackled with linear algebra

- Teaching a computer to read handwriting (and many other achievements of machine learning).
- Modelling the long term evolution of an ecosystem, based on knowledge of how its species interact.
- Sending information across noisy channels, in a manner that ensures that errors can be detected and maybe corrected, if not avoided. This area of mathematics is known as coding theory. Much of it is closely related to linear algebra over finite fields such as the integers modulo a prime.

What is a matrix?

A $m \times n$ matrix over $\mathbb{R}$ (the field of real numbers) is an array of m rows and n columns, whose entries are elements of $\mathbb{R}$. The expression $m \times n$ is referred to as the size of a matrix (even though what it really describes is the shape).

Notation We write $M_{m\times n}(\mathbb{R})$ for the set of all $m \times n$ matrices over $\mathbb{R}$. When $m = n$, we abbreviate this to $M_n(\mathbb{R})$. We can write $M_{m\times n}(\mathbb{C})$ or $M_{m\times n}(\mathbb{Q})$ etc, for sets of matrices with entries in other number systems.

Example Elements of $M_{2\times 3}(\mathbb{R})$ include

$$\begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \end{bmatrix}, \quad \begin{bmatrix} \pi & -\pi & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Why are matrices so important? Reason 1.

> Because they provide an efficient and computationally convenient means of recording essential information in many contexts.

This is one of many answers to the question in the header (we will see more).
We'll demonstrate this first one by looking at Gaussian elimination, a fundamental method for solving systems of linear equations.
We'll study this method through some examples.
It goes back at least to work of Gauss in 1810, and continues to be a basis for modern computational methods. The key innovation is to record the essential information from the system in a matrix, and simplify the matrix in a systematic way.

Consider the equation

$$2x + y = 3.$$

This is an example of a *linear equation* in the variables x and y. As it stands, the statement "$2x + y = 3$" is neither true nor untrue : it is just a statement involving the abstract symbols x and y. However if we replace x and y with some particular pair of real numbers, the statement will become either true or false.

Definition A pair (x_0, y_0) of real numbers is a solution to the equation $2x + y = 3$ if setting $x = x_0$ and $y = y_0$ makes the equation a true statement.

The set of all solutions to the equation is called its *solution set*.

Solutions of Linear Systems

A collection of linear equations in the same n variables is referred to as a *linear system* or *system of linear equations*. The solution set of the system is the subset of $\mathbb{R}^n$ consisting of those elements that satisfy all of the equations in the system.

Example Solve the linear system

$$\begin{array}{rcrcrl} 2x & + & y & = & 3 & \text{(A)} \\ 4x & + & 3y & = & 4 & \text{(B)} \end{array}$$

We can do this with an "ad hoc" approach. This is harder for a more complicated system with more variables, and/or more equations. We will devise a systematic approach, known as Gauss-Jordan elimination.

Example 1

Find all solutions of the following system :

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$$

We can find solutions by simplifying the system through operations of the following types :

1. We can multiply one equation by a non-zero constant.
2. We can add one equation to another (for example in the hope of eliminating a variable from the result).

We now develop a new technique both for describing our system and for applying operations of the above types more systematically and with greater clarity.

The Augmented Matrix

We associate a *matrix* to our system of equations.

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \leftrightarrow \left[\begin{array}{rrr|r} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right] \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array}$$

Definition The above matrix is called the *augmented matrix* of the system of equations. We work with the augmented matrix instead directly with the equations, and allow operations of the following types.

1. Multiply a *row* by a non-zero constant.
2. Add a multiple of one row to another row.
3. Interchange two rows in the matrix

Definition Operations of these three types are called Elementary Row Operations (EROs) on a matrix.

Gaussian Elimination

ERO	Matrix	System
	$\left[\begin{array}{rrrr} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right]$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$
1. $R3 \to R3 + R1$	$\left[\begin{array}{rrrr} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ 0 & 6 & 1 & 5 \end{array}\right]$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ & & 6y & + & z & = & 5 \end{array}$
2. $R2 \to R2 - 3R1$	$\left[\begin{array}{rrrr} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 6 & 1 & 5 \end{array}\right]$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & - & 5y & + & z & = & -6 \\ & & 6y & + & z & = & 5 \end{array}$

ERO	Matrix	System
		$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$
		$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & -11z & = & 11 \end{array}$
		$\begin{array}{rcrcrcrl} x & + & 2y & - & z & = & 5 & (A) \\ & & y & + & 2z & = & -1 & (B) \\ & & & & z & = & -1 & (C) \end{array}$

Backsubstitution

We have produced a new system of equations. This is easily solved.

$$\begin{array}{rcrcrcrl} x & + & 2y & - & z & = & 5 & (A) \\ & & y & + & 2z & = & -1 & (B) \\ & & & & z & = & -1 & (C) \end{array}$$

$$\text{Backsubstitution: } \begin{array}{lll} (C) & z = -1 & \\ (B) & y = -1 - 2z & \implies \quad y = -1 - 2(-1) = 1 \\ (A) & x = 5 - 2y + z & \implies \quad x = 5 - 2(1) + (-1) = 2 \end{array}$$

Solution : $x = 2,\ y = 1, z = -1$

Check that this is a solution of the original system. It is the only solution both of the final system and of the original one (and every intermediate one).

The Row Echelon Form

The matrix obtained in Step 5 above is in *Row-Echelon Form*.

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This means :

1. The first non-zero entry in each row is a 1 (called a *Leading 1*).
2. If a column contains a leading 1, then every entry of the column below the leading 1 is a zero.
3. As we move downwards through the rows of the matrix, the leading 1's move from left to right.
4. Any rows consisting entirely of zeroes are grouped together at the bottom of the matrix.

The process by which the augmented matrix of a system of equations is reduced to row-echelon form is called *Gaussian Elimination*.

General Strategy to Obtain a Row-Echelon Form

1. Get a 1 as the top left entry of the matrix.
2. Use this first leading 1 to "clear out" the rest of the first column, by adding suitable multiples of Row 1 to subsequent rows.
3. If column 2 contains non-zero entries (other than in the first row), use ERO's to get a 1 as the second entry of Row 2. Now use this second leading 1 to "clear out" the rest of column 2 (below Row 2) by adding suitable multiples of Row 2 to subsequent rows.
4. Now go to column 3. If it has non-zero entries (other than in the first two rows) get a 1 as the third entry of Row 3. Use this third leading 1 to clear out the rest of Column 3, then proceed to column 4. Continue until a row-echelon form is obtained.

Definition A matrix is in reduced row-echelon form (RREF) if

1. It is in row-echelon form, and
2. If a particular column contains a leading 1, then *all* other entries of that column are zeros.

If we have a row-echelon form, we can use EROs to obtain a reduced row-echelon form (this is called Gauss-Jordan elimination).

Example To get a RREF from a REF:

$$\begin{bmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Remark Different sequences of ERO's on a matrix can lead to different row-echelon forms. However, the reduced row-echelon form of any matrix is unique.

Leading Variables and Free Variables

Example Find all solutions of the following system :

$$\begin{array}{rcrcrcrcrl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \quad \mathbf{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \quad \mathbf{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \quad \mathbf{III} \end{array}$$

Step 1 Write down the augmented matrix of the system :

$$\left[\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array} \right]$$

Step 2 Use Gauss-Jordan elimination to find a reduced row-echelon form from this augmented matrix.

$$\text{RREF}: \quad \left(\begin{array}{rrrrr} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array} \right)$$
$$x_1 \quad x_2 \quad x_3 \quad x_4$$

Leading variables and free variables (continued)

This matrix corresponds to a new system of equations:

$$\begin{array}{rcrcrl} x_1 & + & 2x_4 & = & 4 & \text{(A)} \implies x_1 = 4 - 2x_4 \\ x_2 & - & x_4 & = & 2 & \text{(B)} \implies x_2 = 2 + x_4 \\ x_3 & + & x_4 & = & 2 & \text{(C)} \implies x_3 = 2 - x_4 \end{array}$$

Definition The variables whose columns in the RREF contain leading 1s (x_1, x_2, x_3) are leading variables. A variable whose column in the RREF does not contain a leading 1 (x_4 in this example) is a free variable.

The RREF tells us how the values of the leading variables x_1, x_2 and x_3 depend on that of the free variable x_4 in a solution. The free variable x_4 may assume the value of *any* real number. The number of solutions is *infinite*.

The general solution is described by assigning a parameter name to the value of each free variable in a solution. In this example

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t);\ t \in \mathbb{R}.$$

Consistent and Inconsistent Systems (Section 1.3.4)

Example Consider the following system :

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array}$$

To find solutions, obtain a row-echelon form from the augmented matrix :

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consistent and Inconsistent Systems

The system of equations corresponding to this REF has as its third equation

$$0x + 0y + 0z = 1$$

This equation clearly has no solutions - no assignment of numerical values to x, y and z will make the value of the expression $0x + 0y + 0z$ equal to anything but zero. Hence the system has no solution.

Definition A system of linear equations is called inconsistent if it has no solution. A system which has a solution is called *consistent*.