Matrix addition and multiplication by scalars

Two matrices can be added together if they have the same size; in this case their sum is obtained by just adding the entries in each position.

The $m \times n$ zero matrix is the $m \times n$ matrix whose entries are all zeros. It is the identity element for addition of $m \times n$ matrices - this means that addition it to another $m \times n$ matrix has no effect.

A matrix can be multiplied by a scalar; this means multiplying each of its entries by that scalar. With these operations of addition and scalar multiplication, the set of $m \times n$ matrices over a field $\mathbb{F}$ is a *vector space* over $\mathbb{F}$.

A vector space is (more or less) an algebraic structure whose elements can be added, subtracted and multiplied by scalars, subject to some compatibility conditions.

Matrix Multiplication I

We can sometimes also *multiply* matrices.

Definition

A column vector is a matrix with one column. A row vector is a matrix with one row.

Definition

Let A be a $m \times n$ matrix and let v be a column vector with n entries. Then the matrix-vector product Av is the column vector obtained by taking the linear combination of the columns of A whose coefficients are the entries of v. It is a column vector with m entries.

Example

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Linear combinations

Definition

Suppose that $v_1, v_2, \dots, v_k$ are elements of a vector space V. A linear combination of $v_1, \dots, v_k$ is an element of V that has the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k,$$

where the a_i are scalars. In this situation the a_i are called the coefficients in the linear combination.

The term *linear comibination* is very intrinsic to the language of linear algebra, we need to understand it well.

Question Which of the following are $\mathbb{R}$-linear combinations of the row vectors $[1\ -2\ 2]$ and $[4\ 0\ 1]$?

(a) $[-1\ -6\ 5]$ (b) $[2\ 4\ 0]$

Matrix multplication II

Definition

Let A and B be matrices of size $m \times p$ and $p \times n$ respectively. Write $v_1, \dots, v_n$ for the columns of B. Then the product AB is the $m \times n$ matrices whose columns are $Av_1, \dots, Av_n$.

Matrix products are often described by their individual entries.
Suppose that A is a $m \times p$ matrix and B is a $p \times n$ matrix.
The entry in Row i and Column j of A is denoted A_{ij}.
The entry in the the (i,j)-position of AB (i.e. Row i and Column j) is the ith entry of the vector Av_j, where the vector v_j is Column j of B. This is the linear combination of the ith entries of the columns of A (i.e. the entries of Row i of A, with coefficients from Column j of B). It is given by

$$(AB)_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \sum_{k=1}^{p} A_{ik}B_{kj}.$$

Matrix multiplication and the scalar product

The expression for $(AB)_{ij}$ above involves the *scalar product* of two vectors with p entries. For a field $\mathbb{F}$, we write $\mathbb{F}^p$ for the vector space of all vectors with p entries from $\mathbb{F}$.

Definition

Let $u = (a_1, \dots, a_p)$ and $v = (b_1, \dots, b_p)$ be vectors in $\mathbb{F}^p$. Then the ordinary scalar product or dot product of u and v is the element of $\mathbb{F}$ defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_p b_p = \sum_{k=1}^{p} a_k b_k .$$

If $u \cdot v = 0$, we say that u and v are *orthogonal* with respect to the scalar product. If $\mathbb{F} = \mathbb{R}$, this means that the vectors u and v are perpendicular in Euclidean space.

Matrix multiplication and the scalar product, II

If A is $m \times p$ with rows $u_1, \dots, u_m$, and B is $p \times n$ with columns $v_1, \dots, v_n$, then the product AB is a table of values of scalar products of Rows of A with Columns of B.

$$AB = \begin{pmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \dots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \dots & u_2 \cdot v_n \\ \vdots & \vdots & & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \dots & u_m \cdot v_n \end{pmatrix}$$

Why are matrices so important? Reason 2

Because their algebraic operations encode interactions and relationships between quantities, that turn up in many contexts, not only within mathematics.

Our next example highlights how matrix multiplication can be hidden in very ordinary calculations that arise in everyday activities - where the matrices involved are just tables of numbers coming from some practical situation.
This is Example 1.3.7 from the lecture notes - see Weekly Challenge 1.

Making sense of matrix multplication

Example Let A be the 3×3 matrix formed by the table that gives the numbers of first year Humanities (H), Engineering (E) and Science (S) students in first year at Eigen University, in 2015, 2016 and 2017.

	H	E	S
2015	50	100	70
2016	60	80	80
2017	80	70	70

$$A = \begin{pmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{pmatrix}$$

Every first year student at Eigen University takes either Linear Algebra (LA) or Calculus (C) or both. The table below shows the numbers of ECTS credits completed annually in each, by students in each of the three subject areas.

	LA	C
H	10	0
E	15	15
S	20	10

$$B = \begin{pmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{pmatrix}$$

Now look at the meaning of the entries of the product AB.