Lecture 3: Example (not in the lecture notes)

1. Solve the following linear system.

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array}$$

Step 1: Reduce the augmented matrix to RREF.

$$\begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2: Identify leading variables (x_1, x_2) and free variablse x_3, x_4, and write the general solution.

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (2 + t - 3s, 1 - 2t + 4s, t, s) : \ t, s \in \mathbb{R} \\ &= (2, 1, 0, 0) + t(1, -2, 1, 0) + s(-3, 4, 0, 1) : \ t, s \in \mathbb{R}. \end{aligned}$$

Example (Part 2)

2. Solve the following linear system.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & 1 \end{array}$$

We must describe all simultaneous solutions of the first three equations that also satisfy the fourth. A solution of the first three has the form

$$(x_1, x_2, x_3, x_4) = (2 + t - 3s, 1 - 2t + 4s, t, s),$$

for real numbers t and s. Insert this information into the fourth equation:

$$2(2+t-3s)-(1-2t+4s)-3t+4s = 1 \Longrightarrow 3+t-6s = 1 \Longrightarrow t = -2+6s.$$

The parameters t and s are no longer independently free.

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (2 + (-2 + 6s) - 3s, 1 - 2(-2 + 6s) + 4s, -2 + 6s, s) \\ &= (3s, 5 - 8s, -2 + 6s, s) : \ s \in \mathbb{R} \\ &= (0, 5, -2, 0) + s(3, -8, 6, 1) : \ s \in \mathbb{R}. \end{aligned}$$

Example (Part 3)

3. Solve the following linear system.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & 1 \\ 3x_1 & - & 2x_2 & - & 2x_3 & - & 5x_4 & = & 10 \end{array}$$

Simultaneous solutions of the first four equations have the form

$$(x_1, x_2, x_3, x_4) = (3s, 5 - 8s, -2 + 6s, s) : \ s \in \mathbb{R}.$$

Check for values of s for which this also satisfies Equation 5:

$$3(3s) - 2(5 - 8s) - 2(-2 + 6s) - 5s = -6 + 8s = 10 \Longrightarrow 8s = 6, \ s = 2.$$

Unique solution: $(x_1, x_2, x_3, x_4) = (6, -11, 10, 2)$.

Example (Part 4)

3. Show that the following linear system is inconsistent.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & 1 \\ 3x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & 3 \end{array}$$

Simultaneous solutions of the first four equations have the form

$$(x_1, x_2, x_3, x_4) = (3s, 5 - 8s, -2 + 6s, s) : \ s \in \mathbb{R}.$$

Check for values of s for which this also satisfies Equation 5:

$$3(3s) + 2(5 - 8s) + 2(-2 + 6s) - 5s = 6 \neq 3.$$

No simultaneous solution of the first four equations also satisfies the last one, the system is inconsistent.

Matrix addition and multiplication by scalars

Two matrices can be added together if they have the same size; in this case their sum is obtained by just adding the entries in each position.

The $m \times n$ zero matrix is the $m \times n$ matrix whose entries are all zeros. It is the identity element for addition of $m \times n$ matrices - this means that addition it to another $m \times n$ matrix has no effect.

A matrix can be multiplied by a scalar; this means multiplying each of its entries by that scalar. With these operations of addition and scalar multiplication, the set of $m \times n$ matrices over a field $\mathbb{F}$ is a *vector space* over $\mathbb{F}$.

A vector space is (more or less) an algebraic structure whose elements can be added, subtracted and multiplied by scalars, subject to some compatibility conditions.

Matrix Multiplication I

We can sometimes also *multiply* matrices.

Definition 2

A *column vector* is a matrix with one column. A *row vector* is a matrix with one row.

Definition 3

Let A be a $m \times n$ matrix and let v be a column vector with n entries. Then the matrix-vector product Av is the column vector obtained by taking the linear combination of the columns of A whose coefficients are the entries of v. It is a column vector with m entries.

Example

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$