Chapter 2

Spanning sets, linear independence and bases

2.1 Subspaces and spanning sets

We think of the real number line $\mathbb{R}$ as begin "1-dimensional", and of $\mathbb{R}^2$ as being "2-dimensional" and of $\mathbb{R}^3$ as being 3-dimensional. These terms are used not only in mathematics but in everyday language aswell. In linear algebra, they mean something quite precise, that also applies much more generally. To say that $\mathbb{R}$ is 1-dimensional means that we only need one real number to specify the position of a point in $\mathbb{R}$. For a point in $\mathbb{R}^2$, we need to specify two real numbers, for example its x and y coordinates - but these are not the only options. We could use its distance from the origin, and the angle that the line segment joining it to the orgin makes with the positive X-axis. We could specify its position relative to another pair of lines, instead of the two coordinate axes. A 2-dimensional space can be viewed as 1-dimensional space augmented with another independent copy of itself. The x-coordinate of a point in $\mathbb{R}^2$ can be any real number. But once that has been specified - say it is 2 - then the range of possibilities for the point includes *all* points in $\mathbb{R}^2$ that lie on the line $x = 2$, all points with x-coordinate 2. The set of such points forms another copy of $\mathbb{R}$ inside $\mathbb{R}^2$, it is the vertical line through $(2, 0)$.

Before we proceed further, it's time to note the official definition of a vector space. We've described a vector space as a set of elements that can be added, subtracted and multiplied by scalars from a *field* such as $\mathbb{R}$. We also require that these algebraic operations interact in sensible ways, as detailed below in the definition.

Definition 2.1.1. *A vector space V over a $\mathbb{R}$ is a non-empty set equipped with an addition operation $(+)$, and whose elements can be multiplied by scalars in $\mathbb{R}$, subject to the following axioms.*

1. *For all $u, v \in V$, $u + v = v + u$ (addition is commutative).*
2. *For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$ (addition is associative).*
3. *V includes an element 0_V, with the property that $0_V + v = v$ for all $v \in V$ (zero element of V or zero vector).*
4. *For every $v \in V$, there exists an element $-v$ of V, with the property that $v + (-v) = 0_V$ (subtraction).*
5. *If $\alpha, \beta \in \mathbb{R}$ and $v \in V$, then $\alpha(\beta v) = \alpha\beta(v)$ (compatibility of scalar multiplication with multiplication in $\mathbb{R}$).*
6. *If $\alpha \in \mathbb{R}$ and $u, v \in V$, then $\alpha(u + v) = \alpha u + \alpha v$ (distributivity of scalar mulatiplication over addition in V).*
7. *If $\alpha, \beta \in \mathbb{R}$ and $v \in V$, then $(\alpha + \beta)v = \alpha v + \beta v$ (distributivity of scalar multiplication over addition in $\mathbb{R}$).*
8. *$1v = v$ for all $v \in V$.*

The field of scalars $\mathbb{R}$ may be replaced with any other field in this definition. A field is a system of scalars or numbers in which we can add, muliply, subtract, and divide by non-zero elements, subject to some axioms that are consistent with the arithmetic of $\mathbb{R}$.

Examples of vector spaces over $\mathbb{R}$

1. The space $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices with real entries (for fixed m and n).

2. The space of all polynomials with real coefficients.

3. The set of complex numbers is a vector space over $\mathbb{R}$.

Besides $\mathbb{R}^2$, another example of a vector space that is 2-dimensional is the space V consisting of all symmetric 2×2 matrices in $M_2(\mathbb{R})$ with trace zero. A symmetric matrix is one that is equal to its transpose. Trace zero means the sum of the entries on the main diagoanl is zero. So a symmetric matrix of trace zero in $M_2(\mathbb{R})$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, where a and b may be any real numbers and may be chosen independently. So it takes a choice of two real numbers to specify an element of V. Also,

$$V = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\} = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

We say that the set $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is a *spanning set* of V over $\mathbb{R}$.

Definition 2.1.2. *Let V be a vector space over a field $\mathbb{F}$. A subset U of V is a* subspace *(or* vector subspace*) of V if U is itself a vector space over $\mathbb{F}$, under the addition and scalar multiplication operations of V.*

Two things need to be checked to confirm that a subset U of a vector space V is a *subspace*:

1. That U is *closed* under the addition in V: that $u_1 + u_2 \in U$ whenever $u_1 \in U$ and $u_2 \in U$;

2. That U is *closed* under scalar multiplication: that $\alpha u \in U$ whenever $u \in U$ and $\alpha \in \mathbb{F}$.

Examples

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are rational polynomials of degree at most 2, then so also is $f(x) + g(x)$. If $f(x)$ is a rational polynomial of degree at least 2, then so is $\alpha f(x)$ for any $\alpha \in \mathbb{Q}$.

2. The set of $\mathbb{C}$ complex numbers is a vector space over the set of real numbers. Within $\mathbb{C}$, the subset $\mathbb{R}$ is an example of a vector subspace over $\mathbb{R}$. An example of a subset of $\mathbb{C}$ that is *not* a real vector subset is the unit circle S in the complex plane - this is the set of complex numbers of modulus 1, it consists of all complex numbers of the form $a + bi$, where $a^2 + b^2 = 1$. This is closed neither under additon nor multiplication by real scalars.

3. The Cartesian plane $\mathbb{R}^2$ is a real vector space. Within $\mathbb{R}^2$, let $U = \{(a, b) : a \geqslant 0, b \geqslant 0\}$. Then U is closed under addition and under multiplication by positive scalars. It is not a vector subspace of $\mathbb{R}^2$, because it is not closed under multiplication by negative scalars.

4. Let v be a (fixed) non-zero vector in $\mathbb{R}^3$, and let $v^\perp = \{u \in \mathbb{R}^3 : u^T v = 0\}$. Then $v^\perp$ is not empty since $0 \in v^\perp$. Suppose that $u_1, u_2 \in v^\perp$. Then $(u_1 + u_2)^T v = (u_1^T + u_2^T)v = u_1^T v + u_2^T v = 0$. So $u_1 + u_2 \in v^\perp$ and $v^\perp$ is closed under addition. If $u \in v^\perp$ and $\alpha \in \mathbb{R}$, then $(\alpha u)^T v = \alpha u^T v = \alpha 0 = 0$, and $\alpha u \in v^\perp$. Hence $v^\perp$ is closed under scalar multiplication in $\mathbb{R}^3$. Conclusion: $v^\perp$ is a vector subspace of $\mathbb{R}^3$. Note that $v^\perp$ is not all of $\mathbb{R}^3$, since $v \notin v^\perp$.

Definition 2.1.3. *Let V be a vector space over a field $\mathbb{F}$, and let S be a non-empty subset of V. The* $\mathbb{F}$-linear span *(or just* span*) of S, denoted $\langle S \rangle$ is the set of all $\mathbb{F}$-linear combinations of elements of S in V. If $S = V$, then S is called a* spanning set *of V. This means that every element of V is a linear combination of elements of S.*

For a subset S of a F-vector space V, the sum of any two linear combinations of elements of S is an element of S, and any scalar multiple of a linear combination of elements of S is again a linear combination of elements of S; hence the following lemma.

Lemma 2.1.4. *For any subset* S *of a vector space* V, $\langle S\rangle$ *is a subspace of* V.

Examples

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$ (this means that P_2 is itself a vector space under the addition and scalar mutliplication operations of $\mathbb{Q}[x]$). If $S = \{x^2 + 1, x + 1\}$, then
$$\langle S\rangle = \{a(x^2 + 1) + b(x + 1) : a, b \in \mathbb{Q}\} = \{ax^2 + bx + a + b : a, b \in \mathbb{Q}\}.$$
So $\langle S\rangle$ consists of all rational polynomials of degree at most 2, in which the constant coefficient is the sum of the coefficients of x and x^2. For example, $x^2 + 2x + 3 \in \langle S\rangle$ but $x^2 + 2x + 4 \notin \langle S\rangle$. Since $\langle S\rangle$ does not include all elements of P_2, S is not a spanning set of P_2 over $\mathbb{Q}$.

2. The set $S = \left\{\binom{3}{1}, \binom{2}{1}, \binom{1}{-1}\right\}$ is a spanning set of the vector space $\mathbb{R}^2$ of all real column vectors with two entries. If $v = \binom{a}{b} \in \mathbb{R}^2$, we can write v as a linear combination of the elements of S, for example by writing
$$\begin{pmatrix}a\\b\end{pmatrix} = (a+b)\begin{pmatrix}3\\1\end{pmatrix} + (-a-b)\begin{pmatrix}2\\1\end{pmatrix} - b\begin{pmatrix}1\\-1\end{pmatrix}.$$
This is not the only way to do it. We could also write
$$\begin{pmatrix}a\\b\end{pmatrix} = (4a+b)\begin{pmatrix}3\\1\end{pmatrix} + (-5a-b)\begin{pmatrix}2\\1\end{pmatrix} + (-a-b)\begin{pmatrix}1\\-1\end{pmatrix}.$$
We could forget about the third element of S and just write
$$\begin{pmatrix}a\\b\end{pmatrix} = (a+b)\begin{pmatrix}3\\1\end{pmatrix} + (-a)\begin{pmatrix}2\\1\end{pmatrix}.$$
So the three elements of S are not necessary to form a spanning set of $\mathbb{R}^2$. We could span $\mathbb{R}^2$ just with the subset $\{\binom{2}{1}, \binom{3}{1}\}$ of S. We note that $\binom{1}{-1}$ is a $\mathbb{R}$-linear combination of the other two elements of S. If we drop this element from S, we can still recover it in the span of the remaining elements.

The second example above motivates this lemma, which we will explore further in the next section.

Lemma 2.1.5. *Suppose that* $S_1 \subset S$, *where* S *is a subset of a vector space* V. *Then* $\langle S_1\rangle \subseteq \langle S\rangle$, *and* $\langle S_1\rangle = \langle S\rangle$ *if and only if every element of* $S\backslash S_1$ *is a linear combination of elements of* S_1.

We finish this section by noting the distinction between a *finite dimensional* and *infinite dimensional* vector space.

Definition 2.1.6. *A vector space is said to be* finite dimensional *if it has a finite spanning set. A vector space that does not have a finite spanning set is* infinite dimensional.

Two examples of infinite dimensional vector spaces

1. The vector space $\mathbb{R}[x]$ of all polynomials with real coefficients is infinite dimensional. To see this, let S be any finite subset of $\mathbb{R}[x]$ (i.e. a finite set of polynomials). Let x^k be the highest power of x to appear in any element of S. Then no linear combination of elements of S has degree exceeding x^k, so the linear span of S cannot be all of $\mathbb{R}[x]$.

2. The set $\mathbb{R}$ of real numbers is infinite dimensional as a vector space over the field $\mathbb{Q}$ of rational numbers.

2.2 Linear Independence

Definition 2.2.1. *Let* S *be a subset of a vector space* V*, having at least 2 elements. Then* S *is* linearly independent *if no element of* S *is a linear combination of the other elements of* S *(equivalently, if no element of* S *belongs to the span of the other elements of* S*).*

A subset consisting of a single element is linear independent, provided that its unique element is not the zero vector.

To decide if a given set is linearly independent, Definition 2.2.1 is maybe not the most useful formulation, because it requires us to check something separately for each element of S, which could take a lot of work. The following altenative version of the definition might have less appeal at an intuitive level, but it is often more useful in practice.

Definition 2.2.2. *Let* S *be a non-empty subset of a vector space* V*. Then* S *is* linearly independent *if the only way to write the zero vector in* V *as a linear combination of elements of* S *is to take all the coefficients to be 0.*

Equivalence of the two definitions
Let $S = \{v_1, \dots, v_k\}$ and suppose that $v_1 \in \langle v_2, \dots, v_k \rangle$. Then

$$v_1 = a_2 v_2 + \cdots + a_k v_k,$$

and

$$0 = -v_1 + a_2 v_2 + \cdots + a_k v_k$$

is an expression for the zero vector as a linear combination of elements of S, whose coefficients are not all zero.

On the other hand, suppose that

$$0 = c_1 v_1 + \cdots + c_k v_k,$$

where the scalars c_i are not all zero. If $c_1 \neq 0$ (for example), then the above equation can be rearranged to express v_1 as a linear combination of $v_2, \dots, v_k$:

$$v_1 = -\frac{c_2}{c_1} v_2 - \cdots - \frac{c_k}{c_1} v_k.$$

Example 2.2.3. *In* $\mathbb{R}^3$*, let* $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} \right\}$.

To determine whether S is linearly independent, we must investigate whether the system of equations

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solutions other than $(x, y, z) = (0, 0, 0)$. The augmented matrix of this system, and its RREF, are

$$\begin{bmatrix} 1 & -2 & -3 & 0 \\ 2 & 3 & 8 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus for any t, $(x, y, z) = (-t, -2t, t)$ is a solution, and for example by taking $t = 1$ we see that

$$-1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and hence that each of the three elements of S is a linear combination of the other two. So S is not linearly independent (we say that S is *linearly dependent*).

We note below some characterizations of linearly independent set. Let S be a subset of a vector space V.

1. S is linearly independent if S is a *minimal* spanning set of its linear span - no proper subset of S spans the same subspace of V that S does, or every proper subspace of S spans a strictly smaller subspace than S itself.

2. S is linearly independent if every element of $\langle S \rangle$ has a *unique* expression as a linear combination of elements of S. If a particular element of $\langle S \rangle$ had two different expressions as linear combinations of S, with different coefficients, then subtracting one from the other would give a non-trivial expression for 0_V as a linear combination of elements of S, and by Definition 2.2.2 we would conclude that S is linearly dependent.

3. Another version of 2. above: S is linearly independent if every element of the span of S has *unique coordinates* in terms of the elements of S.

So a linearly independent set in a vector space V is a *minimal* or *irredundant* spanning set for its linear span. If its linear span happens to be all of V, it gets a special name.

Definition 2.2.4. *A* basis *of a vector space* V *is a spanning set of* V *that is linearly independent. [Plural: bases]*

Lemma 2.2.5. *If* S *is a finite spanning set of a vector space* V*, then* S *contains a basis of* V*.*

Proof. If S is not linearly independent, then some element v_1 of S is in the span of the other elements of S, and $S_1 := S \backslash \{v_1\}$ is again a spanning set of V. If S_1 is not linearly independent, then we can discard an element of S_1 that is in the linear span of the others, to form a smaller spanning set S_2. Since S is finite, this process cannot continue indefinitely, and it concludes with a linearly independent spanning set of V. □

We will show that if V has a finite basis, then *every* basis has the same number of elements. This number is then referred to as the *dimension* of V. The key to this is to show that the number of elements in *any* spanning set of V is an upper bound for the number of elements in *any* linearly independent subset of V. The following theorem

Theorem 2.2.6. *Let* V *be a vector space over a field* $\mathbb{F}$*, and suppose that* $S = \{v_1, \ldots, v_n\}$ *is a spanning set of* V*. Then the number of elements in a linearly independent subset of* V *cannot exceed* n*.*

Proof. Let $S = \{v_1, \ldots, v_t\}$ be a *spanning set* of V. Let $L = \{y_1, \ldots, y_k\}$ be a linearly independent subset of V. We need to show $k \leqslant t$.

1. We know that $y_1 \in V$, so y_1 is a linear combination of elements of S. After reordering the elements of S if necessary, we may assume that v_1 appears with a non-zero coefficient in an expression for y_1 as a linear combination of $v_1, \ldots, v_n$. It follows that v_1 belongs to the linear span of $\{y_1, v_2 \ldots, v_t\}$, and hence that
$$S_1 = \{y_1, v_2, \ldots, v_t\}$$
is again a spanning set of V.

2. Then y_2 is a linear combination of the elements of S_1, and it is not just a scalar multiple of y_1, since L is linearly independent. So one of $v_2, \ldots, v_t$ is involved (with non-zero coefficient) in an expression for y_2 as a linear combination of $y_1, v_2, \ldots, v_t$. After relabelling if necessary, we can suppose that v_2 has this property. Then v_2 is a linear combination of $y_1, y_2, v_3, \ldots, v_n$, and
$$S_2 = \{y_1, y_2, v_3, \ldots, v_t\}$$
is a spanning set of V.

3. Continuing in this way (and relabelling the elements of S) at each step if necessary, we adjust the spanning set by replacing v_3 with y_3, then v_4 wth y_4, and so on. Since $\{v_1, \ldots, v_{k-1}\}$ is not a spanning set of V, there is still at least one element of the orignal S left after $k-1$ steps, at which point we have $S_{k-1} = \{v_1, v_2, \ldots, v_{k-1}, y_k, \ldots, y_t\}$.

We conclude that $t \geqslant k$, and the number of elements in a linearly independent subset of V cannot exceed the number in a spanning set. □

Theorem 2.2.6 is often referred to as the *Steinitz exchange lemma*, and it is the most important technical ingredient in establishing the following crucial property of finite dimensional vector spaces V.

Theorem 2.2.7. *If V is a finite dimensional vector space over a field $\mathbb{F}$, then every basis of V has the same number of elements.*

Proof. Let B_1 and B_2 be bases of V. Then B_1 is linearly independent and B_2 is a spanning set of V, so $|B_1| \leqslant |B_2|$ by Theorem 2.2.6. Also, B_2 is linearly independent and B_1 is a spanning set of V, so $|B_2| \leqslant |B_1|$ by Theorem 2.2.6. Hence $|B_1| = |B_2|$. □

Definition 2.2.8. *The number of elements in any (hence every) basis of a finite dimensional vector space V is called the* dimension *of V, denoted* $\dim V$.

Example 2.2.9. Let V be the space of skew-symmetric matrices in $M_3(\mathbb{R})$ (a matrix A is *skew-symmetric* if $A^T = -A$). Then

$$V = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The typical element of V noted above can be written as

$$\begin{aligned} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} &= a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}), \end{aligned}$$

where we write E_{ij} for the matrix with 1 in the (i, j)-position and zero in all other positions. From the above, we see that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a spanning set of V. This set is also linearly independent: if

$$a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}) = 0_{3\times 3}$$

for some real scalars a, b, c, then by looking at the entries in the $(1, 2), (1, 3)$ and $(2, 3)$ positions, we observe that $a = b = c = 0$. We conclude that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a basis of V and that $\dim V = 3$.

Exercise 2.2.10. *What is the dimension of the space of skew-symmetric matrices in $M_n(\mathbb{R})$? What about the space of symmetric matrices?*

2.3 More on Bases and Dimension

In this section we note a few more properties of bases of finite dimensional vector spaces. We start with a connection to matrices. We let V be a vector space of dimension n over a field $\mathbb{F}$. Recall that this means that every basis of V has n elements, and a basis of V is a linearly independent spanning set of B. This means that it is both a minimal spanning set of V and a maximal linearly independent subset of V.

Lemma 2.3.1. *Every linearly independent subset of V with n elements is a basis of V.*

Proof. Let $L = \{v_1, \ldots, v_n\}$ be a linearly independent subset of V. If L is not a spanning set of V, then there is some $v \in V$ with $v \notin \langle L \rangle$. It follows that the set $L' = \{v_1, \ldots, v_n, v\}$ is linearly independent in V, contrary to Theorem 2.2.6. □

Lemma 2.3.2. *Every spanning set of V with n elements is a basis of V.*

Proof. Let V be a spanning set of V with n elements. If V is not linearly independent, then V contains a proper subset that spans V but has fewer than n elements, contrary to Theorem 2.2.6. □

Lemma 2.3.3. *If L is a linearly independent subset of V, then L can be extended to a basis of V.*

Proof. Write $L = \{v_1, \dots, v_k\}$. Then $k \leqslant n$ by Theorem 2.2.6. If $k = n$, then L spans V by Lemma 2.3.1, and L is a basis of V. If $k < n$, then L is not a spaning set of V, and there is an element $v_{k+1} \in V$ with $v_{k+1} \notin \langle L \rangle$. Then $\{v_1, \dots, v_k, v_{k+1}\}$ is linearly independent. We may continue in this way to add elements from outside the existing span, until we reach a basis of V after $n - k$ steps. □

For any field $\mathbb{F}$, $\mathbb{F}^n$ denotes the space of all column vectors with n entries. The *standard basis* of $\mathbb{F}^n$ is $\{e_1, \dots, e_n\}$, where e_i has 1 in position i and 0 in all other positions. That $\{e_1, \dots, e_n\}$ is linearly independent and spans $\mathbb{F}^n$ can be confirmed from the relevant definitions.

We finish (for now) on this topic by noting that $\mathbb{F}^n$ is the generic and even (sort of) the only vector space of dimension n over $\mathbb{F}$. Suppose that V is a $\mathbb{F}$-vector space with $\dim V = n$, and let $B = \{v_1, \dots, v_n\}$ be a basis of V over $\mathbb{F}$. For every element $v \in V$, there is a unique expression for v as a linear combination of the elements of B:

$$v = a_1 v_1 + \cdots + a_n v_n.$$

We refer to $a_1, \dots, a_n$ as the *coordinates* of v with respect to the basis B. With this association, we can consider v to be represented by the column vector in $\mathbb{F}^n$ whose entries are $a_1, \dots, a_n$. This association defines a bijective correspondence between V and $\mathbb{F}^n$, and means that we can identify these two vector spaces as being essentially the same. Different bases of V correspond to different identifications of V with $\mathbb{F}^n$, and we will explore the relationships between these in Chapter 3.