4.3 Quotient Groups

Let N be a normal subgroup of the group G. The theme of this section is to define a group structure on the set of left (or right) cosets of N in G, which is denoted by G/N.

To motivate this definition we give a relatively familiar example.

Example 4.3.1. ***(Addition modulo 5)***

Let $5\mathbb{Z}$ denote the subgroup of $(\mathbb{Z}, +)$ consisting of all multiples of 5.

$$5\mathbb{Z} = \{\dots, -10, -5, 0, 5, 10, 15, \dots\}.$$

There are five distinct cosets of $5\mathbb{Z}$ in $\mathbb{Z}$, as follows:

$$\begin{aligned} 5\mathbb{Z} &= \{\dots, -10, -5, 0, 5, 10, 15, \dots\}; \\ 1 + 5\mathbb{Z} &= \{\dots, -9, -4, 1, 6, 11, 16, \dots\}; \\ 2 + 5\mathbb{Z} &= \{\dots, -8, -3, 2, 7, 12, 17, \dots\}; \\ 3 + 5\mathbb{Z} &= \{\dots, -7, -2, 3, 8, 13, 18, \dots\}; \\ 4 + 5\mathbb{Z} &= \{\dots, -6, -1, 4, 9, 14, 19, \dots\}. \end{aligned}$$

Note that $5 + 5\mathbb{Z} = 5\mathbb{Z}$, $6 + 5\mathbb{Z} = 1 + 5\mathbb{Z} = 11 + 5\mathbb{Z}$, etc. We give the names $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$ to the five cosets above, write $\mathbb{Z}/5\mathbb{Z}$ for the set consisting of these cosets, and define addition in $\mathbb{Z}/5\mathbb{Z}$ as in the following example:

To add $\bar{3}$ and $\bar{4}$, we choose a representative from each of these cosets, add the representatives in $\mathbb{Z}$, and then take the coset to which the result belongs. For example we could take 3 and 4 as our representatives, add them together to get 7, notice that 7 belongs to the coset $\bar{2}$ and conclude that $\bar{3} + \bar{4} = \bar{2}$.
Alternatively we could take 8 and -11 as our representatives of $\bar{3}$ and $\bar{4}$, adding these in $\mathbb{Z}$ would give -3 which again belongs to $\bar{2}$, so again we would conclude $\bar{3} + \bar{4} = \bar{2}$.
The key point is that *the outcome does not depend on the choice of coset representatives*, and this is because $5\mathbb{Z}$ is a *normal* subgroup of $\mathbb{Z}$.

Let G be a group and let N be a normal subgroup of G. Let G/N (read this as "G mod N") denote the set of (left or right) cosets of N in G. Define an operation $\star$ on G/N by

$$xN \star yN = xyN,$$

where $x, y \in G$. This is basically saying that to "multiply" two cosets of N in G, we should take an element from each one, multiply them in G and then take the coset determined by the result.

We need to show that the operation $\star$ is well-defined in the following sense: if x, x_1, y, y_2 are elements of G for which $xN = x_1N$ and $yN = y_1N$, then we want to know that $xN \star yN = x_1N \star y_1N$, i.e. that $xyN = x_1y_1N$.

- Since $xN = x_1N$ we know that $x^{-1}x_1 \in N$; write $x^{-1}x_1 = n_x$.
- Since $yN = y_1N$ we know that $y^{-1}y_1 \in N$: write $y^{-1}y_1 = n_y$.
- What we need to do in order to show that $xyN = x_1y_1N$ is show that $(xy)^{-1}x_1y_1 \in N$, i.e. that $y^{-1}x^{-1}x_1y_1 \in N$. Now $y^{-1}x^{-1}x_1y_1 = y^{-1}n_xy_1$. Note that $y^{-1}n_x$ is in the left coset $y^{-1}N$. Then it is in the right coset Ny^{-1}, because N is normal in G. Then $y^{-1}n_x = ny^{-1}$ for some $n \in N$ and

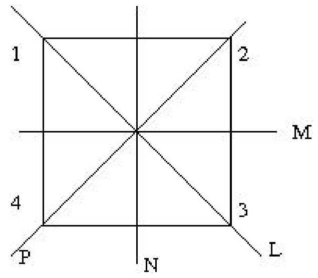
  $$(xy)^{-1}x_1y_1 = y^{-1}n_xy_1 = ny^{-1}y_1 = nn_y = nn_x \in N.$$

  This means that $xyN = x_1y_1N$, as required.

Now $\star$ defines a binary operation on G/N, the set of cosets of N in G. The coset N itself is an identity element for this operation. The operation $\star$ is associative because it is based on the associative operation of G. The inverse of the coset gN under $\star$ is the coset $g^{-1}N$. Thus G/N becomes a group under $\star$, called the *quotient group* G mod N.

Note: If G is finite, then the order of G/N is $[G : N] = \dfrac{|G|}{|N|}$.

Example 4.3.2. *Suppose that* D_8 *is the group of symmetries of the square (with axes as in the diagram below) and that* N *is the subgroup* $\{\mathrm{id}, R_{180}\}$.



Then N *is a normal subgroup of* D_8 *and it has four cosets:*

$$N = \{\mathrm{id}, R_{180}\},\ R_{90}N = \{R_{90}, R_{270}\},\ S_L N = \{S_L, S_P\},\ S_M N = \{S_M, S_N\}.$$

The multiplication table for the quotient group D_8/N *is as follows.*

	N	$R_{90}N$	$S_L N$	$S_M N$
N	N	$R_{90}N$	$S_L N$	$S_M N$
$R_{90}N$	$R_{90}N$	N	$S_M N$	$S_L N$
$S_L N$	$S_L N$	$S_M N$	N	$R_{90}N$
$S_M N$	$S_M N$	$S_L N$	$R_{90}N$	N