4.2 Normal Subgroups

Suppose that $\phi : G \to H$ is a homomorphism of groups, and let N denote the kernel of ϕ. So N consists of all those elements x of G for which $\phi(x) = \mathrm{id}_H$. In particular the elements of N all have the same image under ϕ.

Now let g be an element of G and suppose that $g \notin N$. Write h for the image of g under ϕ, so $h = \phi(y)$ and $h \neq \mathrm{id}_H$ in H. Now consider the set of all elements of G whose image under ϕ is h. How is this related to y and to N?

- Let $n \in N$. Then $\phi(gn) = \phi(g)\phi(n) = h\mathrm{id}_H = h$. This means that every element of the left coset gN of N in G has image h.
- On the other hand suppose that $\phi(g') = h$ for some $g' \in G$. Then $g' = g(g^{-1}g')$ and $g^{-1}g' \in N$ since
$$\phi(g^{-1}g') = (\phi(g))^{-1}\phi(g') = h^{-1}h = \mathrm{id}_H.$$
Hence $g' \in gN$.

Thus, if $\phi(g) = h$, then the set of elements of G whose image under ϕ is h is exactly the left coset gN.

A very similar argument shows that the set of elements of G whose image under ϕ is h is exactly the *right coset* Ng.

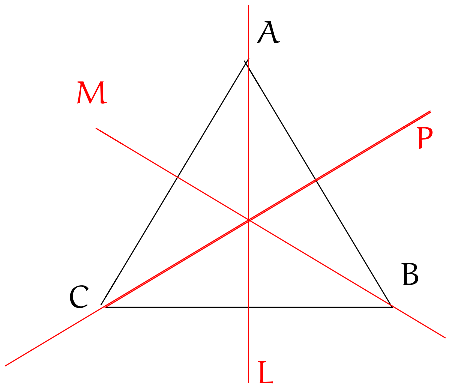
- If $n \in N$, then $\phi(ng) = \phi(n)\phi(g) = \mathrm{id}_H h = h$.
- Also, if $\phi(g') = h$ for some $g' \in G$ then $g' = (g'g^{-1})g$ and $g'g^{-1} \in N$ since $\phi(g'g^{-1}) = hh^{-1} = \mathrm{id}_H$. So g' belongs to the right coset Ng of N in G.

The conclusion from the above discussion is:
If $\phi : G \to H$ is a group homomorphism with kernel N, and if $g \in G$, then the subset of G consisting of those elements whose image under ϕ is the same as that of g is equal to both the left coset gN and the right coset Ng. In particular, for every $g \in G$, the left coset gN and the right coset Ng are equal to each other as sets.

Definition 4.2.1. *Let* N *be a subgroup of a group* G. *If* $gN = Ng$ *for every* $g \in G$, *then* N *is said to be a* normal subgroup *of* G. *This situtaion is denoted* $N \trianglelefteq G$.

EXAMPLES OF NORMAL AND NON-NORMAL SUBGROUPS

Consider the group D_6 of symmetries of the equilateral triangle, labelled $\mathrm{id}, R_{120}, R_{240}, T_L, T_M, T_P$, with axes M, L, P as in the diagram below.



The table for D_6 is as follows.

	id	R_{120}	R_{240}	T_L	T_M	T_P
id	id	R_{120}	R_{240}	T_L	T_M	T_P
R_{120}	R_{120}	R_{240}	id	T_M	T_P	T_L
R_{240}	R_{240}	id	R_{120}	T_P	T_L	T_M
T_L	T_L	T_P	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_P	R_{120}	id	R_{240}
T_P	T_P	T_M	T_L	R_{240}	R_{120}	id

Let H denote the subgroup $\{\mathrm{id}, T_L\}$ of order 2 of D_6. To determine whether H is a normal subgroup of D_6 or not, look at the left and right cosets of H in D_6 determined by each element. For example

$$T_P H = \{T_P \circ \mathrm{id}, T_P \circ T_L\} = \{T_P, R_{240}\}.$$

On the other hand

$$HT_P = \{\mathrm{id} \circ T_P, T_L \circ T_P\} = \{T_P, R_{120}\}.$$

Since the right and left cosets of H determined by the element T_P are different sets, we can say that H is *not* a normal subgroup of D_6.

Let N denote the subgroup of D_6 consisting of the three rotations id, R_{120} and R_{240}. For each element of D_6 we can investigate whether the left and right cosets of N that they determine coincide or not.

- In the case of the elements id, R_{120} and R_{240} of N, the left and right cosets of N that these determine are all equal to N itself and in particular all equal to each other.
- The left coset of N determined by T_L is

  $$\{T_L \circ \mathrm{id}, T_L \circ R_{120}, T_L \circ R_{240}\} = \{T_L, T_P, T_M\}.$$

  The right coset of N determined by T_L is

  $$\{\mathrm{id} \circ T_L, R_{120} \circ T_L, R_{240} \circ T_L\} = \{T_L, T_M, T_P\}.$$

  Thus $T_L N = NT_L$.
  You can check that it is also true for the other two reflections that $T_M N = NT_M$ and $T_P N = NT_P$. Having checked all of these we can state that N is a *normal subgroup* of D_6.

Other characterizations of normality

Our orignal definition of normality was that a subgroup N of a group G is *normal* in G if and only if for every $g \in G$, the left coset gN is equal to the right coset Ng. This does not have to mean that $gn = ng$ for all $g \in G$ and for all $n \in N$ (although that is a possibility that arises if N is in the centre of G). Note that

- In an abelian group, all subgroups are normal.
- In any group G, all subgroups of the centre $Z(G)$ are normal in G.

Suppose that N is a normal subgroup of a group G. This means that for every $g \in G$, every element of the left coset gN also belongs to the right coset Ng. This means that for all $g \in G$ and all $n \in N$,

$$gn = n'g$$

for some $n' \in N$. Multiplying the above equation (on the right) by g^{-1}, this means that

$$gng^{-1} \in N, \ \forall g \in G, \text{ and } \forall n \in N.$$

This is saying that a normal subgroup must be closed under conjugation in the whole group: *if* N *is a normal subgroup of* G *and* $n \in N$*, then all conjugates of* n *in* G *belong to* N.

On the other hand, if a subgroup of G has the above property, then it *is* normal, as the following lemma shows.

Lemma 4.2.2. *Let* N *be a subgroup of a group* G*. Then* N *is normal in* G *if and only if* $gng^{-1} \in N$ *for all* $g \in G$ *and for all* $n \in N$*.*

Proof. If $N \trianglelefteq G$, then the above observations show that $gng^{-1} \in N$ for all $n \in N$ and all $g \in G$. On the other hand, assume that this condition holds and let $g \in G$. To show that $gN \subseteq Ng$, let gn be a typical element of the left coset gN (so $n \in N$). Then $gng^{-1} = n'$ for some $n' \in N$, and $gn = n'g$ which means that gn belongs to the right coset Ng. Thus $gN \subseteq Ng$.

For the opposite inclusion, let ng be a typcial element of the right coset Ng (so $n \in N$). Note that $g^{-1}ng = g^{-1}n(g^{-1})^{-1} \in N$, so $g^{-1}ng = n''$ for some $n'' \in N$. Then $ng = gn''$, so ng belongs to the left coset gN. Hence $Ng \subseteq gN$, and we conclude that $gN = Ng$ for all $g \in G$, which means that N is normal in G. □

The following (which are just alternative wordings of the same statement) are useful characterizations of normal subgroups that are equivalent to our original definition, and may equally well be used as a definition.

1. A subgroup N of a group G is *normal* in G if $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$.
2. A subgroup N of G is *normal* in G if N is closed under conjugation in G.
3. A subgroup N of G is *normal* in G if N is a union of conjugacy classes of G (and is a subgroup of course).

Looking back at our examples from D_6, we can interpret them in the context of the above statements.
We noted that the group $N = \{\mathrm{id}, R_{120}, R_{240}\}$ of rotations *is* a normal subgroup of G; this is the union of the conjugacy classes $\{\mathrm{id}\}$ and $\{R_{120}, R_{240}\}$.
We noted that the subgroup $H = \{\mathrm{id}, T_L\}$ is *not* normal in D_6. We can explain this now by observing that this subgroup contains T_L but does not contain the elements T_M and T_P which are conjugate to T_L in D_6 - thus H is not closed under conjugation in D_6.

We finish this section by noting that with our new understanding of normality in terms of conjugacy, it is easy to show that the kernel of a group homomorphism must be a normal subgroup.

Lemma 4.2.3. *Let* $\phi : G \to H$ *be a group homomorphism with kernel* N*. Then* N *is a normal subgroup of* G*.*

Proof. We know from Lemma 4.1.3 that N is a subgroup of G. To see that it is normal, let $n \in N$ and $g \in G$. We need to show that $gng^{-1} \in N$, which means that $\phi(gng^{-1}) = \mathrm{id}_H$. We can see this as follows

$$\begin{aligned}
\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g^{-1}) \\
&= \phi(g)\,\mathrm{id}_H\,\phi(g^{-1}) \\
&= \phi(g)\phi(g^{-1}) \\
&= \phi(gg^{-1}) \\
&= \phi(\mathrm{id}_G) \\
&= \mathrm{id}_H.
\end{aligned}$$

Thus $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$, and $N \trianglelefteq G$.

□