Lecture 19: Normal Subgroups

Definition A subgroup N of a group G is **normal** in G ($N \trianglelefteq G$) if for every $g \in G$ the left coset gN is equal to the right coset Ng.

Equivalent Definition A subgroup N is **normal** in G if $gng^{-1} \in N$ for every $n \in N$ and $g \in G$. This means N is a union of conjugacy classes of G, as N contains all G-conjugates of each of its elements.

Version 1$\Longrightarrow$ Version 2 Suppose that $gN = Ng$ for all $g \in G$, and let $n \in N$. Then $gn \in Ng$, so $gn = n'g$ for some $n' \in N$. Thus $gng^{-1} \in N$, and the G-conjugacy class of n is contained in N.

Version 2$\Longrightarrow$ Version 1 Suppose that N is a union of conjugacy classes, and let $g \in G$. Then, for each $n \in N$,
$gng^{-1} \in N \Longrightarrow gn = n'g$ for some $n' \in N$, so $gn \in Ng$. Thus $gN \subseteq Ng$, and the same argument applied to $g^{-1}ng$ shows that $gN \subseteq Ng$.

Examples of Normal Subgroups

- Every group is a normal subgroup of itself, and the trivial subgroup is a normal subgroup of every group.
- Every subgroup of an abelian group is normal (conjugacy classes in abelian groups are single elements).
- Any subgroup that is the kernel of a group homomorphism is normal.
- In D_6 the subgroup consisting of the three rotations is normal, and the subgroup consisting of any one reflection is non-normal.
- Any subgroup of index 2 in any group is normal.

Kernels of group homomorphisms are normal subgroups

Let $\phi : G \to H$ be a homomorphism of groups.
Write N for the kernel of ϕ, so $N = \ker\phi = \{x \in G : \phi(x) = \mathrm{id}_H\}$.
Recall from Week 10 that $\ker\phi$ is a subgroup of G.

Claim $N \trianglelefteq G$.
Proof Let $n \in N$. We must show that $g \star_G n \star_G g^{-1} \in N$ for all $g \in G$. So choose $g \in G$. Then

$$\begin{aligned} \phi(g \star_G n \star_G g^{-1}) &= \phi(g) \star_H \phi(n) \star_H \phi(g^{-1}) \\ &= \phi(g) \star_H \mathrm{id}_H \star_H \phi(g^{-1}) \\ &= \phi(g) \star_H \phi(g^{-1}) \\ &= \phi(g \star_G g^{-1}) \\ &= \phi(\mathrm{id}_G) = \mathrm{id}_H \end{aligned}$$

So $g \star_G n \star_G g^{-1} \in N$ for all $n \in N$ and $g \in G$, and the kernel of a group homomorphism is always a normal subgroup of the domain.

Lecture 22: Quotient Groups

Let $\phi : G \to H$ be a group homomorphism with kernel N.
We saw in Week 10 that the distinct elements of $\operatorname{Im} \phi$ correspond exactly to the distinct cosets of N in G: two elements of G have the same image under ϕ if and only if they belong to the same (left or right) coset of N in G.

This means that the group operation in $\operatorname{Im} \phi$ (or in H) can be interpreted as a binary operation on the set of cosets of N in G, with respect to which this set is a group.

To multiply two cosets of N in G: take elements x and y respectively from each coset. Take the element $x \star_G y$ in G. The coset to which this element belongs is the product of the orginal two cosets in G/N, the quotient group of G modulo N.

The outcome does not depend on the initial choice of x and y from the two cosets.

Quotient Groups

This can be formulated without the context of a homomorphism.

Let N be a normal subgroup of a group G. Define a multiplication $\star$ on the set G/N of cosets of N in G by $xN \star yN = xyN$. This is well-defined: suppose $xN = x_1N$ and $yN = y_1N$. This means $x^{-1}x_1 \in N$ and $y^{-1}y_1 \in N$. Write $x^{-1}x_1 = n_x$ and $y^{-1}y_1 = n_y$. We need to know $xyN = x_1y_1N$, i.e. that $(xy)^{-1}x_1y_1 \in N$.

$$(xy)^{-1}x_1y_1 = y^{-1}x^{-1}x_1y_1 = y^{-1}\underbrace{x^{-1}x_1}_{n_x \in N}y = y^{-1}n_xy$$

Since $N \trianglelefteq G$ and $n_x \in N$, the element $y^{-1}n_x$ belongs to the right coset Ny^{-1}, i.e. $y^{-1}n_x = ny^{-1}$ for some $n \in N$. Hence

$$(xy)^{-1}x_1y_1 = y^{-1}n_xy_1 = ny^{-1}y_1 = nn_y \in N,$$

and the multiplication operation on cosets is well-defined.

Quotient Groups (continued)

If G is a group with a normal subgroup N, we define a multiplication $\star$ on the set G/N of cosets of N in G by $xN \star yN = xyN$.

- This multiplication is well-defined as shown in the last slide (this depends on the normality of N in G).
- This multiplication operation is associative since it comes from the associative operation of G;
- G/N has N $(= \mathrm{id}\, N)$ as an identity element;
- the element xN of G/N has $x^{-1}N$ as its inverse.

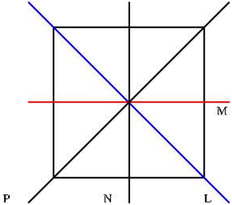
So G/N is a group, called the quotient group G modulo N.

Examples

1. The subgroup $5\mathbb{Z}$ of the group of integers under addition is the group consisting of all multiples of 5. The cosets of $5\mathbb{Z}$ in $\mathbb{Z}$ are the congruence classes of integers modulo 5,

   $$\bar{0} = 5\mathbb{Z},\ \bar{1} = 1 + 5\mathbb{Z},\ \bar{2} = 2 + 5\mathbb{Z},\ \bar{3} = 3 + 5\mathbb{Z},\ \bar{4} = 4 + 5\mathbb{Z}.$$

   The quotient group $\mathbb{Z}/5\mathbb{Z}$ is the additive group of integers modulo 5. The elements are the classes above, and addition is defined by adding representatives of classes modulo 5.
2. The subgroup $N = \{\mathrm{id}, \mathbb{R}_{180}\}$ of D_8 is normal. There are four cosets of N in D_8. The group table for D_8/N is below



	N	$R_{90}N$	T_LN	T_MN
N	N	$R_{90}N$	T_LN	T_MN
$R_{90}N$	$R_{90}N$	N	T_MN	T_LN
T_LN	T_LN	T_MN	N	$R_{90}N$
T_MN	T_MN	T_LN	$R_{90}N$	N