6. Partition: 4+1
   Cycle type: one 4-cycle and one fixed point
   Representative of class: (1 2 3 4)
   No. of elements in class: $\binom{5}{4} \times 3! = 5 \times 6 = 30$
   Order of centralizer of an element of this class: $\frac{120}{30} = 4$

   Note on Count: We have $\binom{5}{4}$ choices for the four elements to be in our 4-cycle, and having chosen them there are 3! ways to arrange them in cyclic order. For example if our elements are 1,2,3,4 we can agree to write 1 first in our description of the cyclic order, we have 3 choices for what to put next, 2 after that and so on.

7. Partition: 5
   Cycle type: one 5-cycle
   Representative of class: (1 2 3 4 5)
   No. of elements in class: $4! = 24$
   Order of centralizer of an element of this class: $\frac{120}{24} = 5$

So the number of conjugacy classes of S_5 is 7. We should find that our numbers of elements in each add up to 120:

$$1 + 10 + 15 + 20 + 20 + 30 + 24 = 120.$$

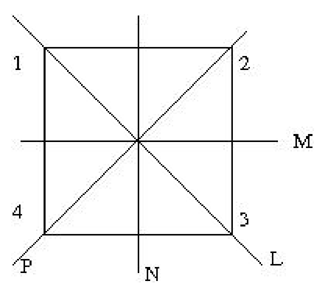
Note: We have shown that the centre of S_n (for $n \geqslant 3$) is trivial, since the centre consists exactly of those elements that have only one element in their conjugacy class. Every cycle type except the one with n fixed points is represented by more than one element.

The symmetric groups are exceptional in that their conjugacy classes have a nice combinatorial description. This is not really typical of finite groups.

3.2 Examples

The goal of this section is to explain the concept of a group action, along with the related concepts of orbit and stabilizer, via a selection of examples. A *group action* occurs when every element of a group determines a permutation of some set.

1. Let $G = D_8$, the symmetry group of the square, with vertices and axes of symmetry labelled as follows. The elements of G are transformations of the whole square, but we can consider them specifically as permutations of the vertices.



   With the vertices labelled 1,2,3,4 as in this picture, the elements of D_8 respectively correspond to permutations of $\{1, 2, 3, 4\}$ as follows.

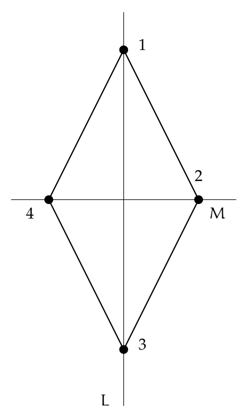
id $\leftrightarrow$ id	$T_M \leftrightarrow (1\,4)(2\,3)$
$R_{90} \leftrightarrow (1\,4\,3\,2)$	$T_N \leftrightarrow (1\,2)(3\,4)$
$R_{180} \leftrightarrow (1\,3)(2\,4)$	$T_L \leftrightarrow (2\,4)$
$R_{270} \leftrightarrow (1\,2\,3\,4)$	$T_P \leftrightarrow (1\,3)$

   Note that every element of D_8 determines a different permutation of the set $\{1, 2, 3, 4\}$ of vertices of the square - one of these is the identity, two are 4-cycles, three are pairs of disjoint transpositions and two are single transpositions. Not every permutation of $\{1, 2, 3, 4\}$ arises as a symmetry of the square - for example the permutation $(1\ 2)$ does not. However the description above amounts to an identification of D_8 with a particular subgroup of order 8 of the symmetric group S_4 (you can check that the composition operation on D_8 matches that on the group of associated permutations).

We say that D_8 *acts* on the set $\{1,2,3,4\}$ of vertices of the square. This means that every element of D_8 determines a permutation of $\{1,2,3,4\}$, that the permutation determined by the identity element of D_8 is the identity permutation, and that the permutation determined by the composition of two elements of D_8 is the composition in S_4 of the two permutations that they determine separately.

Choose any vertex of the square, for example 2. For each of the four vertices 1,2,3,4, there is an element of D_8 that takes 2 to that vertex. We say that *the orbit of the element 2 under the action of* G *is the full set* $\{1,2,3,4\}$ *of vertices*. For any particular vertex, we can identify the elements of G that leave this vertex fixed. This set is called the *stabilizer* of the element and it is always a subgroup of G. In our example the stabilizer of both 1 and 3 is the set $\{\mathrm{id}, T_L\}$ and the stabilizer of both 2 and 4 is $\{\mathrm{id}, T_P\}$.

2. Let G be the group of symmetries of the diamond shown below. Then G has four elements - the identity, the rotation through 180° and the reflections in the axes L and M.



Here again, the elements of G correspond to permutations of the set of vertices as follows.

$$\begin{array}{ll} \mathrm{id} \leftrightarrow \mathrm{id} & T_M \leftrightarrow (1\,3) \\ R_{180} \leftrightarrow (1\,3)(2\,4) & T_L \leftrightarrow (2\,4) \end{array}$$

In this case every element of the group takes the vertex 1 either to 1 or 3, so the *orbit* of 1 is just $\{1,3\}$. Every element of the group takes the vertex 2 either to 2 or 4, so the orbit of 2 is the set $\{2,4\}$. This is an example of a group action with *two orbits*.
The stabilizers of the various elements are given by

$$\mathrm{Stab}(1) = \{\mathrm{id}, T_L\},\ \mathrm{Stab}(2) = \{\mathrm{id}, T_M\},\ \mathrm{Stab}(3) = \{\mathrm{id}, T_L\},\ \mathrm{Stab}(4) = \{\mathrm{id}, T_M\}.$$

In both of these examples it can be observed for each vertex that the product of the number of elements in its stablizer and the number of elements in its orbit is the group order (this is the Orbit-Stabilizer theorem).

3. Let $G = \{1, -1, i, -i\}$, a group under multiplication. Mutiplying by any element of G determines an invertible function from $\mathbb{C}$ to $\mathbb{C}$. Multiplication by 1 is the identity function on $\mathbb{C}$. Multiplication by i rotates a point through 90° counterclockwise about zero; multiplication by -1 and $-i$ similary act as rotations of the complex plane, through 180° and 270° respectively.

For a nonzero complex number z, the orbit of z under this action consists of the four vertices of the square that is centered at zero and has z as one vertex. The stabilizer of every such z is the trivial subgroup of G. The orbit of the complex number 0 consists only of 0, and its stabilizer is all of G.

4. Consider the group $GL(3,\mathbb{R})$ of invertible 3×3 matrices over $\mathbb{R}$. Since every matrix in $GL(3,\mathbb{R})$ determines a linear transformation of $\mathbb{R}^3$, we can think of G as the group of all invertible linear tranformations of $\mathbb{R}^3$. We say that G *acts on the set* $\mathbb{R}^3$. For a column vector v in $\mathbb{R}^3$ and an invertible matrix $A \in GL(3,\mathbb{R})$, the element A takes v to Av (also an element of $\mathbb{R}^3$).

   If v is a non-zero vector in $\mathbb{R}^3$ and if u is another one, then there is some invertible matrix A for which $Av = u$, so u and v belong to the same *orbit* under the action of $GL(3, R)$. The stabilizer of v is the set of invertible matrices A for which $Av = v$, which is a subgroup of $GL(3,\mathbb{R})$.

   The zero vector is its own image under left multiplication by any matrix, so it forms an orbit by itself, and its stablizer is all of $GL(3,\mathbb{R})$. This is an example of an action of an infinite group on an infinite set, with two orbits.

5. Let G be a finite group with elements $\{g_1, g_2, \ldots, g_n\}$. Let $x \in G$ (so x is one of the g_i). The function that takes g_i to xg_i is a permutation of the elements of G determined by the element x. So every element of G determines a permutation of the set G by left multiplication. This is an action of G on the set of its own elements. There is a single orbit and the stabilizer of each element is trivial.

6. Again let G be a finite group with elements $\{g_1, g_2, \ldots, g_n\}$. Let $x \in G$ (so x is one of the g_i). Another permutation of G that is defined by the element x is the one given by

$$g_i \to x g_i x^{-1}.$$

   Under this action, the *orbit* of any element is exactly its conjugacy class, and the stabilizer of an element is exactly its centralizer in G.