Chapter 3

Group Actions

3.1 Conjugacy in symmetric groups

Definition 3.1.1. *The group consisting of all permutations of a set of* n *elements is called the* symmetric group of degree n *and denoted* S_n.

REMARKS

1. The order of S_n is $n!$, the number of permutations of n objects (read this as "n factorial").
2. We often think of the n elements being permuted as the first n positive integers $1, 2, \ldots, n$, but this is not intrinsic to the definition of S_n. It doesn't really matter what these elements are called as long as they have distinct labels.
3. Although the terminology is potentially problematic, it is important not to confuse the term "symmetric group" with groups of symmetries of (for example) regular polygons.

This section starts with a reminder of how to represent permutations and how to do calculations with them. We then give a combinatorial description of conjugacy classes in symmetric groups.

An element of S_4 is a permutation of the set $\{1, 2, 3, 4\}$; this means a function from that set to itself that sends each element to a different image, and hence shuffles the four elements. In S_4, a basic way to represent the permutation $1 \to 1,\ 2 \to 4,\ 3 \to 2,\ 4 \to 3$ is by the array

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

Representing permutations like this we can practise multiplying (or composing) them. In these notes we will use the convention that for permutations σ and τ, the product $\sigma\tau$ means "σ after τ or $\sigma \circ \tau$, i.e. that the factor that is written on the right is applied first. This is not a universally agreed convention and people use both possible interpretations. For this course it is probably a good idea that we all share the same interpretation to avoid confusion, but in general all that is important is that you state in which order you are considering the composition to take place and that you are consistent.

Example 3.1.2. *In* S_5*, suppose that*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix},\ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}.$$

Calculate the products $\sigma\tau$ *and* $\tau\sigma$.

Solution: To calculate $\sigma\tau$, we apply τ first and then σ. Remember that this is just a composition of functions.

- τ sends 1 to 4, then σ sends 4 to 4. So $\sigma\tau$ sends 1 to 4.

- τ sends 2 to 2, then σ sends 2 to 3. So $\sigma\tau$ sends 2 to 3.
- τ sends 3 to 3, then σ sends 3 to 5. So $\sigma\tau$ sends 3 to 5.
- τ sends 4 to 5, then σ sends 5 to 1. So $\sigma\tau$ sends 4 to 1.
- τ sends 5 to 1, then σ sends 1 to 2. So $\sigma\tau$ sends 5 to 2.

We conclude that

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$$

You would not be expected to provide all this detail in every example like this, it is provided here to explain how the process works. It's a good idea to practise this so that you can do the calculation in one line. The answer to the second part is

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

This array format is not the only way of representing a permutation and not always the most useful way. Another way of thinking about a permutation π is by thinking about how it moves the elements of the set around, by starting with a single element and looking at the sequence of images when you repeatedly apply π to it. Eventually you will have to get back to the original element. Consider the following example in S_{14}.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 9 & 8 & 2 & 5 & 1 & 12 & 14 & 6 & 7 & 3 & 13 & 10 & 4 \end{pmatrix}$$

Start with the element 1 and look at what happens to it when you repeatedly apply π.

- First you get $1 \to 11$;
- Then $11 \to 3$;
- Then $3 \to 8$;
- Then $8 \to 14$;
- Then $14 \to 4$;
- Then $4 \to 2$;
- Then $2 \to 9$;
- Then $9 \to 6$;
- Then $6 \to 1$.

After nine applications of π we arrive back at 1 and this is the first time we have a repetition in the list. This will happen every time: the list can't continue indefinitely without repetition because there are only finitely many elements being permuted. Suppose that after starting at 1 the first repetition occurs at Step k, after k applications of π. Then we have

$$1 \to a_1 \to a_2 \to \cdots \to a_{k-1} \to$$

where $1, a_1, \dots, a_{k-1}$ are distinct. The next element (a_k) is a repeat of one of these. However it can't be a repeat of a_1, because 1 is the only element whose image under π is a_1, and $a_{k-1} \neq 1$. The same applies to $a_2, \dots, a_{k-1}$. So it must be that 1 (the element where we started) is the first element to be repeated, and that we close the circle that started with 1. In our example above there were nine distinct elements in the sequence that started at 1. So the permutation π produces the following *cycle*:

$$1 \to 11 \to 3 \to 8 \to 14 \to 4 \to 2 \to 9 \to 6 \to 1$$

This cycle is often written using the following notation:

$$(1\ \ 11\ \ 3\ \ 8\ \ 14\ \ 4\ \ 2\ \ 9\ \ 6).$$

Note that 1 is not written at the end here. The above notation means the permutation (of 14 elements in this case) that sends 1 to 11, 11 to 3, etc, and sends 6 back to 1. There is nothing in the notation to indicate that we are talking about an element of S_{14} - this has to be clear from the context. Also, it is understood that elements that are not mentioned in the above notation are fixed by the permutation that it denotes. The permutation $(1\ 11\ 3\ 8\ 14\ 4\ 2\ 9\ 6)$ is an example of a *cycle of length 9* in S_{14}. It is not the same as the permutation π that we started with, but it does coincide with π on the set of nine elements that can be obtained by starting at 1 and repeatedly applying π. This set is called the *orbit* of 1 under π.

The point of this discussion is that π can be written as a product (or composition) of *disjoint* cycles in S_{14}. The next step towards doing so is to look for the first element (in the natural order) of our set that is not involved in the first cycle. This is 5. Go back to π and see what happens to 5 under repeated application of π. We find that

$$5 \to 5,$$

so 5 is fixed by π. We could think of this as a cycle of length 1.

There are still some elements unaccounted for. The first one is 7. Looking at the orbit of 7 under π, we find

$$7 \to 12 \to 13 \to 10 \to 7$$

so we get the cycle $(7\ \ 12\ \ 13\ \ 10)$ of length 4. Note that this has no intersection with the previous cycles.

Our conclusion is that π can be written as the product of these disjoint cycles:

$$\pi = (1\ \ 11\ \ 3\ \ 8\ \ 14\ \ 4\ \ 2\ \ 9\ \ 6)(7\ \ 12\ \ 13\ \ 10).$$

If you like you can explicitly include (5) as a third factor, but the usual convention is not to bother including elements that are fixed in expressions of this nature, if an element does not appear it is understood to be fixed.

Notes

1. The representation of π in "array" format can easily be read from its representation as a product of disjoint cycles. For example if you want to know the image of 8 under π, just look at the cycle where 8 appears - its image under π is the next element that appears after it in that cycle, 14 in this example. If your element is written at the end of a cycle, like 10 in this example, then its image under π is the number that is written in the first position of that same cycle (so $10 \to 7$ here). An element that does not appear in any of the cycles is fixed by the permutation.

2. The statement above says that π can be effected by first applying the cycle $(7\ \ 12\ \ 13\ \ 10)$ (which only moves the elements $7, 12, 13, 10$) and then applying the cycle $(1\ \ 11\ \ 3\ \ 8\ \ 14\ \ 4\ \ 2\ \ 9\ \ 6)$ (which only moves the elements $1, 11, 3, 8, 14, 4, 2, 9, 6$). Since these two cycles operate on disjoint sets of elements and do not interfere with each other, they commute with each other under composition - it does not matter which is written first in the expression for π as a product of the two of them. So we could equally well write

$$\pi = (7\ \ 12\ \ 13\ \ 10)(1\ \ 11\ \ 3\ \ 8\ \ 14\ \ 4\ \ 2\ \ 9\ \ 6).$$

3. The expression for a permutation as a product of disjoint cycles is unique up to the order in which the cycles are written. This means that the same cycles must appear in any such expression for a given permutation, but they can be written in different orders.

   It might also be worth mentioning that a given cycle can be written in slightly different ways, since it doesn't matter which element is taken as the "starting point". For example $(7\ \ 12\ \ 13\ \ 10)$ and $(13\ \ 10\ \ 7\ \ 12)$ represent the same cycle.

Definition 3.1.3. *The expression of an element of* S_n *as a product of disjoint cycles partitions the set* $\{1, 2, \ldots, n\}$ *into disjoint* orbits. *In the above example there are three orbits:*

$$\{1, 2, 3, 4, 6, 8, 9, 11, 14\},\ \{5\},\ \{7, 10, 12, 13\}.$$

If two elements belong to the same orbit for a permutation π, it means that some power of π takes one of those elements to the other. Note that fixed points *do* count as orbits. So the identity element of S_n has n orbits each consisting of a single element. A permutation in S_n has just one orbit if it is a single cycle involving all n elements.

It is good idea to practise moving between the "array representation" and "disjoint cycle representation" of a permutation.

A definition that we have sort of been using but that hasn't been stated yet is that of the *order* of an element of a group. Try not to confuse this with the order of the group itself.

Definition 3.1.4. *Let* G *be a group and let* $g \in G$. *The* order *of the element* g *is the least positive integer* k *for whcih* g^k *is the identity element. If no such* k *exists,* g *is said to have* infinite order. *The order of* g *is the order of the cyclic subgroup generated by* G.

So, for example, the order of each of the (non-identity) rotations of D_6 is 3, and the order of each reflection in any dihedral group is 2. The order of the identity element is always 1.

In a symmetric group, the order of a *cycle of length* k (also called a k-cycle) is k. This is because the cycle must be applied k times in order to map every element to itself and so obtain the identity permutation. In order to determine the order of a general element of S_n, look at its expression as a product of disjoint cycles. The order of the element is the least common multiple of the lengths of the disjoint cycles that appear in it.

Example 3.1.5. *What is the order of the element*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 2 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}?$$

Solution: Look at the expression for π as a product of disjoint cycles:

$$\pi = (1\ 6)(2\ 4\ 8\ 7\ 3).$$

We see that π is the product of a cycle of length 2 and a cycle of length 5. These disjoint cycles commute with each other which means that $\pi^k = \mathrm{id}$ for an integer k if and only if the kth powers of both the cycle of length 2 and the cycle of length 5 are equal to the identity. In order for the kth power of the transposition $(1\ 6)$ to be the identity, k must be even. In order for the kth power of the 5-cycle $(2\ 4\ 8\ 7\ 3)$ to be the identity, k must be a multiple of 5. We conclude that the order of π is $\mathrm{lcm}(2, 5) = 10$. (Note that $\mathrm{lcm}(2, 5)$ means the least common multiple of 2 and 5).

One nice feature of the symmetric groups is that their conjugacy classes are easy to describe. We will say that two elements of S_n have the *same cycle type* if, when written as products of disjoint cycles, they both involve the same number of 1-cycles, the same number of 2-cycles, the same number of 3-cycles, and so on. For example, in S_{12}, the permutations

$$(1\ 4\ 3\ 5\ 11)(7\ 8\ 9) \text{ and } (2\ 8\ 7\ 12\ 3)(1\ 11\ 9)$$

both have the same cycle structure. Each of them involves one 5-cycle, one 3-cycle and four fixed points (1-cycles).

Theorem 3.1.6. *Let* $\pi = (a_1\ a_2 \ldots\ a_k)$ *be a cycle of length* k *in* S_n*, and let* σ *be any permutation in* S_n. *Then the conjugate* $\sigma\pi\sigma^{-1}$ *is the cycle* $(\sigma(a_1)\ \sigma(a_2) \ldots \sigma(a_k))$.

Proof. (more or less) We will see how this works for the particular example where $n = 7$, $k = 5$, $\pi = (1\ 2\ 3\ 4\ 5)$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 3 & 5 & 1 & 4 \end{pmatrix}$.

We want to consider the permutation $\sigma\pi\sigma^{-1}$. The elements 1 and 4 are sent by σ^{-1} to 6 and 7, which are not moved by π, and then mapped respectively back to 1 and 4 by σ.

So 1 and 4, which are the images under σ of the fixed points of π, are fixed points of $\sigma\pi\sigma^{-1}$.

Now look at what happens to $2, 7, 6, 3, 5$ which are, respectively, the images under σ of the elements $1, 2, 3, 4, 5$ that are cycled (in that order) by π. First, σ^{-1} sends $2, 7, 6, 3, 5$ to $1, 2, 3, 4, 5$ respectively. Then π cycles these around, sending the list $1, 2, 3, 4, 5$ to $2, 3, 4, 5, 1$. Then σ maps the list $2, 3, 4, 5, 1$ back to $7, 6, 3, 5, 2$. So overall, the element $\sigma\pi\sigma^{-1}$ sends $2 \to 7$, $7 \to 6$, $6 \to 3$, $3 \to 5$ and $5 \to 2$. Thus this element is the cycle $(2\ 7\ 6\ 3\ 5)$, which is exactly $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \sigma(4)\ \sigma(5))$, where $\pi = (1\ 2\ 3\ 4\ 5)$. □

Theorem 3.1.6 has the following important consequence.

Theorem 3.1.7. *Let π be any permutation in S_n. Then every conjugate of π in S_n has the same cycle type has π.*

Proof. Let $\pi_1, \dots, \pi_k$ be the disjoint cycles in π, and suppose that σ is an element of S_n and we want to look at the conjugate $\sigma\pi\sigma^{-1}$ of π. Now

$$\pi = \pi_1\pi_2 \dots \pi_k$$

and

$$\begin{aligned} \sigma\pi\sigma^{-1} &= \sigma\pi_1\pi_2 \dots \pi_k\sigma^{-1} \\ &= \sigma\pi_1\sigma^{-1}\,\sigma\pi_2\sigma^{-1} \dots \sigma\pi_k\sigma^{-1}. \end{aligned}$$

By Theorem 3.1.6, $\sigma\pi_i\sigma^{-1}$ is the cycle of the same length as π_i, that cycles the images under σ of the elements that are cycled by π_i. Since the images under σ of the disjoint orbits of π are still disjoint,

$$\sigma\pi_1\sigma^{-1}\,\sigma\pi_2\sigma^{-1} \dots \sigma\pi_k\sigma^{-1}$$

is exactly the expression for $\sigma\pi\sigma^{-1}$ as a product of disjoint cycles. It has the same cycle type as π, since for each i, $\sigma\pi_i\sigma^{-1}$ is a cycle of the same length as π_i. □

The last part of this story is that if two elements of S_n have the same cycle type, then they *are* conjugate to each other in S_n. Theorem 3.1.7 and its proof show how to establish this. Again we will do it by example. Suppose you want to show that the elements π_1 and π_2 are conjugate to each other in S_8, where

$$\pi_1 = (1\ 3\ 4)(5\ 6), \quad \pi_2 = (4\ 8\ 7)(1\ 2).$$

Then π_1 and π_2 have the same cycle type obviously. Theorem 3.1.7 says that for any $\sigma \in S_8$,

$$\sigma\pi_1\sigma^{-1} = (\sigma(1)\ \sigma(3)\ \sigma(4))(\sigma(5)\ \sigma(6)).$$

If we want this to be equal to π_2 we should choose σ so that

$$\sigma(1) = 4,\ \sigma(3) = 8,\ \sigma(4) = 7,\ \sigma(5) = 1,\ \sigma(6) = 2.$$

For $\sigma(2)$, $\sigma(7)$ and $\sigma(8)$ we can do whatever we like (amongst the available options). If we choose

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 8 & 7 & 1 & 2 & 5 & 6 \end{pmatrix}$$

Then we have $\sigma\pi_1\sigma^{-1} = \pi_2$, as required.

Our conclusion is the following theorem.

Theorem 3.1.8. *Two elements of S_n are in the same conjugacy class if and only if they have the same cycle type.*

This means that the number of conjugacy classes in S_n is equal to the number of cycle types. This is the number of *partitions* of n. A partition of n is a way of writing n as a sum of positive integers. So for example the partitions of 4 are

$$4 = 1 + 1 + 1 + 1,\ 4 = 2 + 1 + 1,\ 4 = 2 + 2,\ 4 = 3 + 1,\ 4 = 4.$$

So there are 5 partitions of 4, meaning there are five conjugacy classes in S_4. The partition $1 + 1 + 1 + 1$ corresponds to the cycle type with four fixed points, which means the identity permutation.

The partition $2+1+1$ corresponds to the cycle type with one cycle of length 2 and two fixed points, i.e. the transpositions (there are $\binom{4}{2} = 6$ of these in S_4). The partition 4 corresponds to the cycles of length 4, e.g. $(1\,2\,3\,4)$. There are 6 of these in S_4.

Unfortunately there is no neat formula that tells us how many partitions a given positive integer n has. For small values of n however, we can count them. Also, we can count the number of elements in S_n with a given cycle type, so we can count the number of elements in each conjugacy class. Remember also that the number of elements in a conjugacy class of any group is the index of the centralizer of an element of that class. So we can also calculate the orders of the centralizer of an element of each class. This information is all given below for the example of S_5 - and S_6 is on Problem Sheet 3.

CONJUGACY CLASSES OF S_5

The order of S_5 is $5! = 120$.

1. Partition: 1+1+1+1+1
   Cycle type: five fixed points
   Representative of class: id
   No. of elements in class: 1
   Order of centralizer: 120

2. Partition: 2+1+1+1
   Cycle type: one 2-cycle and three fixed points
   Representative of class: $(1\,2)$
   No. of elements in class: $\binom{5}{2} = 10$
   Order of centralizer of an element of this class: $\frac{120}{10} = 12$

3. Partition: 2+2+1
   Cycle type: two disjoint 2-cycles and one fixed point
   Representative of class: $(1\,2)(3\,4)$
   No. of elements in class: $\binom{5}{2} \times \binom{3}{2} \times \frac{1}{2} = 15$
   Order of centralizer of an element of this class: $\frac{120}{15} = 8$

   Note on Count: We have $\binom{5}{2} = 10$ choices for the first transposition and having chosen this we have $\binom{3}{2} = 3$ choices for the second one. This would give $10 \times 3 = 30$ choices for a pair of disjoint transpositions written in a specified order. Since the order doesn't matter (i.e. $(1\,2)(3\,4)$ is the same permutation as $(3\,4)(1\,2)$, this estimate of 30 counts every pair of disjoint transpositions twice. We need to divide it by 2 to get the right number of elements with this cycle type.

4. Partition: 3+1+1
   Cycle type: one 3-cycle and two fixed points
   Representative of class: $(1\,2\,3)$
   No. of elements in class: $\binom{5}{3} \times 2! = 10 \times 2 = 20$
   Order of centralizer of an element of this class: $\frac{120}{20} = 6$

   Note on Count: We have $\binom{5}{3}$ choices for the three elements to put in our cycle. Having chosen them we have 2! ways to arrange them in cyclic order. For example if our three elements are 1,2,3, they can be arranged in cyclic order as $(1\,2\,3)$ or $(1\,3\,2)$.

5. Partition: 3+2
   Cycle type: one 3-cycle and one 2-cycle (disjoint from the 3-cycle)
   Representative of class: $(1\,2\,3)(4\,5)$
   No. of elements in class: $\binom{5}{3} \times 2! = 10 \times 2 = 20$
   Order of centralizer of an element of this class: $\frac{120}{20} = 6$

   Note on Count: This is the same as the previous class, since having chosen the 3-cycle on one of 20 ways we have no choice about the 2-cycle.

6. Partition: 4+1
   Cycle type: one 4-cycle and one fixed point
   Representative of class: $(1\ 2\ 3\ 4)$
   No. of elements in class: $\binom{5}{4} \times 3! = 5 \times 6 = 30$
   Order of centralizer of an element of this class: $\frac{120}{30} = 4$

   Note on Count: We have $\binom{5}{4}$ choices for the four elements to be in our 4-cycle, and having chosen them there are 3! ways to arrange them in cyclic order. For example if our elements are 1,2,3,4 we can agree to write 1 first in our description of the cyclic order, we have 3 choices for what to put next, 2 after that and so on.

7. Partition: 5
   Cycle type: one 5-cycle
   Representative of class: $(1\ 2\ 3\ 4\ 5)$
   No. of elements in class: $4! = 24$
   Order of centralizer of an element of this class: $\frac{120}{24} = 5$

So the number of conjugacy classes of S_5 is 7. We should find that our numbers of elements in each add up to 120:

$$1 + 10 + 15 + 20 + 20 + 30 + 24 = 120.$$

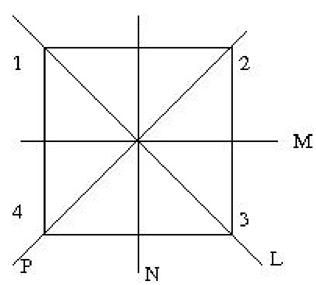
Note: We have shown that the centre of S_n (for $n \geqslant 3$) is trivial, since the centre consists exactly of those elements that have only one element in their conjugacy class. Every cycle type except the one with n fixed points is represented by more than one element.

The symmetric groups are exceptional in that their conjugacy classes have a nice combinatorial description. This is not really typical of finite groups.

3.2 Examples

The goal of this section is to explain the concept of a group action, along with the related concepts of orbit and stabilizer, via a selection of examples. A *group action* occurs when every element of a group determines a permutation of some set.

1. Let $G = D_8$, the symmetry group of the square, with vertices and axes of symmetry labelled as follows. The elements of G are transformations of the whole square, but we can consider them specifically as permutations of the vertices.



   With the vertices labelled 1,2,3,4 as in this picture, the elements of D_8 respectively correspond to permutations of $\{1, 2, 3, 4\}$ as follows.

$\mathrm{id} \leftrightarrow \mathrm{id}$	$T_M \leftrightarrow (1\ 4)(2\ 3)$
$R_{90} \leftrightarrow (1\ 4\ 3\ 2)$	$T_N \leftrightarrow (1\ 2)(3\ 4)$
$R_{180} \leftrightarrow (1\ 3)(2\ 4)$	$T_L \leftrightarrow (2\ 4)$
$R_{270} \leftrightarrow (1\ 2\ 3\ 4)$	$T_P \leftrightarrow (1\ 3)$

   Note that every element of D_8 determines a different permutation of the set $\{1, 2, 3, 4\}$ of vertices of the square - one of these is the identity, two are 4-cycles, three are pairs of disjoint transpositions and two are single transpositions. Not every permutation of $\{1, 2, 3, 4\}$ arises