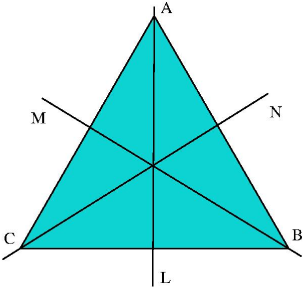
The Centre of Group

Definition Let G be a group with operation $\star$. The *centre* of G, denoted by $Z(G)$ is the subset of G consisting of all those elements that commute with *every* element of G, i.e.

$$Z(G) = \{x \in G : x \star g = g \star x \text{ for all } g \in G\}.$$

Example The centre of D_6 can be read from the group table. It consists only of the identity element.



$\circ$	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

Notes and Examples

- The centre of a group G is equal to G if and only if G is abelian.
- The centre of G is usually denoted by $Z(G)$.
- The identity element always belongs to the centre (this is built into the group axioms).
- The centre of D_{2n} consists only of the identity if n is odd, and also includes the rotation through 180° if n is even.
- The center of the general linear group $\mathrm{GL}(2, \mathbb{R})$ is the set of all invertible scalar matrices, which have the same non-zero entry in both positions on the main diagonal, and zeros in the other positions.

Theorem 2.2.3: $Z(G)$ is a subgroup of G

Proof

1. $\mathrm{id}_G \in Z(G)$, from the group axioms.
2. $Z(G)$ is closed under the operation $\star$ of G:
   Suppose $x, y \in Z(G)$ and let $g \in G$. We must show $(x \star y) \star g = g \star (x \star y)$. We know

$$\begin{aligned}(x \star y) \star g &= x \star (y \star g) = x \star (g \star y) \\ &= (x \star g) \star y = (g \star x) \star y = g \star (x \star y).\end{aligned}$$

3. Suppose $x \in Z(G)$. We must show $x^{-1} \in Z(G)$ also:
   Let $g \in G$. Then

$$\begin{aligned}x \star g = g \star x &\implies x^{-1} \star (x \star g) \star x^{-1} = x^{-1} \star (g \star x) \star x^{-1} \\ &\implies g \star x^{-1} = x^{-1} \star g\end{aligned}$$

   so $x^{-1} \in Z(G)$.

The centralizer of an element

Definition Let G be a group with operation $\star$, and let $x \in G$. The *centralizer* of x in G, denoted by $C_G(x)$, is the subset of G consisting of all those elements of G that commute with x under $\star$,

$$C_G(x) = \{y \in G : x \star y = y \star x\}.$$

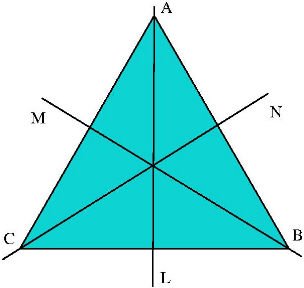
Notes

- $C_G(x)$ is a subgroup of G. (Exercise: prove this)
- It is easy to confuse the concepts of centre and centralizer. The centre belongs to the whole group. A centralizer belongs to a particular element.
- For every element x of G, $Z(G) \subseteq C_G(x)$.
  The centre of G is the intersection of the centralizers of all the elements of G.

Example

Let G be D_6, the symmetry group of the equilateral triangle. We can read the element centralizers from the group table.

- $C_G(\text{id}) = \{\text{id}, R_{120}, R_{240}, T_L, T_M, T_N\} = G$
- $C_G(R_{120}) = \{\text{id}, R_{120}, R_{240}\}$ - the subgroup of rotations
- $C_G(R_{240}) = \{\text{id}, R_{120}, R_{240}\}$ - the subgroup of rotations
- $C_G(T_L) = \{\text{id}, T_L\}$, $C_G(T_M) = \{\text{id}, T_M\}$, $C_G(T_N) = \{\text{id}, T_N\}$.



$\circ$	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

A remark about towers of subgroups

Let G be a non-abelian group (it is only for non-abelian groups that centralizers are of interest).
Let x be an element of G that is not in the centre.
Then

- $C_G(x)$ is a proper subgroup of G.
- $Z(G)$ is a proper subgroup of $C_G(x)$.
- So we have a chain of proper inclusions

$$Z(G) \subsetneq C_G(x) \subsetneq G.$$

If the groups in question are finite, it follows that the index of $Z(G)$ in G cannot be prime.