Lecture 15 and 16: Group Actions (Chapter 3)

Here's that permutation in S_{14} again, and its disjoint cycle representation.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 9 & 8 & 2 & 5 & 1 & 12 & 14 & 6 & 7 & 3 & 13 & 10 & 4 \end{pmatrix}$$

$$\pi = (1 \ \ 11 \ \ 3 \ \ 8 \ \ 14 \ \ 4 \ \ 2 \ \ 9 \ \ 6)(7 \ \ 12 \ \ 13 \ \ 10)(5).$$

There are cycles of length 9, 4 and 1.

Note that π has order $36 = \mathrm{lcm}(9, 4)$, as an element of S_{14}.

So π generates a cyclic subgroup of S_{14}, of order 36.

Let's call this subgroup G.

Action and Orbits

In S_{14}, $\pi = (1\ \ 11\ \ 3\ \ 8\ \ 14\ \ 4\ \ 2\ \ 9\ \ 6)(7\ \ 12\ \ 13\ \ 10)(5)$.

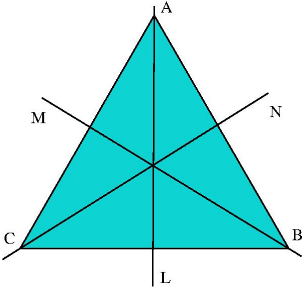
We have already referred to the sets $\{5\}$, $\{7, 10, 12, 13\}$, and $\{1, 2, 3, 4, 6, 8, 9, 11, 14\}$ as the orbits of the permutation π.

From the viewpoint of group actions, we say that the group $G = \langle \pi \rangle$ acts on the set $\{1, 2, \ldots, 14\}$. This means that every element of G permutes the elements of $\{1, 2, \ldots, 14\}$.

We say that the sets $\{5\}$, $\{7, 10, 12, 13\}$, and $\{1, 2, 3, 4, 6, 8, 9, 11, 14\}$ are the orbits of this action.
This means that given elements x and y of $\{1, 2, \ldots, 14\}$, there is an element σ of G that takes x to y if and only if x and y are in the same orbit.

Note: The full symmetric group S_{14} also acts on $\{1, 2, \ldots, 14\}$. For this action there is only one orbit, the full set $\{1, 2, \ldots, 14\}$.

Another example: dihedral groups



D_6 acts on the set $\{A, B, C\}$ of vertices of the triangle.

- $R_{120} \cdot A = C$
- $T_L \cdot A = A$
- $T_M \cdot C = A$

There is a single orbit consisting of all three vertices.
The elements of D_6 that send the vertex A to itself are the identity and the reflection T_L. We say that the stabilizer of A is $\{\mathrm{id}, T_L\}$. Note that this is a subgroup of D_6.

Note also that $2 \times 3 = 6$, where 2 is the order of the stabilizer of A, 3 is the number of elements in the orbit of A, and 6 is the order of the group D_6.

Definition of a Group Action

Definition Let G be a group and let S be a set. We say that G acts on S if every element g of G corresponds to a permutation π_g of S, satisfying the following two conditions:

1. π_{id} is the identity permutation of S.
2. If g and h are elements of G, then $\pi_{gh} = \pi_g \circ \pi_h$.

Notation If $g \in G$ and $x \in S$, it is conventional to write $g \cdot x$ for the element of S obtained by applying the permutation determined by g to x. In this notation, the above conditions are

1. $\mathrm{id} \cdot x = x, \ \forall x \in S$.
2. $g \cdot (h \cdot x) = (gh) \cdot x, \forall x \in S, \ \forall g, h \in G$.

Orbits and Stabilizers

Let G be a group acting on a set S.

Definition For $x \in S$, the orbit of x, denoted $G \cdot x$, is the set of elements of S that are obtained by applying an element of G to x.

$$G \cdot x = \{g \cdot x : g \in G\}.$$

Definition For $x \in S$, the stabilizer of x in G, denoted $\mathrm{Stab}_G(x)$, is the subset of G consisting of those elements that fix x.

$$\mathrm{Stab}_G(x) = \{g \in G : \ g \cdot x = x\}.$$

Two Lemmas

1. For $x, y \in G$, $G \cdot x$ and $G \cdot y$ are either equal or disjoint, and S is the disjoint union of the distinct orbits under the action of G
2. For $x \in S$, $\mathrm{Stab}_G(x)$ is a subgroup of G.

The Stabilizer of an Element

Let $x \in S$. The stabilizer of x, denoted $\mathsf{Stab}_G(x)$, is the set of elements g of G that fix x.

$$\mathsf{Stab}_G(x) = \{g \in G : g \cdot x = x\} \subseteq G.$$

Lemma $\mathsf{Stab}_G(x)$ is a subgroup of G.

Proof

1. $\mathrm{id}_G \in \mathsf{Stab}_G(x)$, since $\mathrm{id}_G \cdot x = x$.
2. Closure: suppose $g, h, \in \mathsf{Stab}_G(x)$. We show $gh \in \mathsf{Stab}_G(x)$.

$$gh \cdot x = g \cdot (h \cdot x) = g \cdot x = x \Longrightarrow gh \in \mathsf{Stab}_G(x).$$

3. Inverse: Suppose $g \in \mathsf{Stab}_G(x)$. We show $g^{-1} \in \mathsf{Stab}_G(x)$.

$$\begin{aligned} g \cdot x = x &\implies g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x \\ &\implies \mathrm{id}_G \cdot x = g^{-1} \cdot x \\ &\implies x = g^{-1} \cdot x. \end{aligned}$$

The Orbit-Stabilizer Theorem

For a finite group G acting on a set S, and for any $x \in S$, the number of elements in the orbit of x is the index in G of the stabilizer of x.

$$|G \cdot x| = [G : \mathrm{Stab}_G(x)]$$

Proof For elements g and h of G, we consider when $g \cdot x$ and $h \cdot x$ are the same (or different) elements of S.

$$\begin{aligned} g \cdot x = h \cdot x &\iff g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (h \cdot x) \\ &\iff \mathrm{id}_G \cdot x = g^{-1} h \cdot x \\ &\iff x = g^{-1} h \cdot x \\ &\iff g^{-1} h \in \mathrm{Stab}_G(x) \\ &\iff h \in g\, \mathrm{Stab}_G(x). \end{aligned}$$

So $g \cdot x = h \cdot x$ if and only if g and h are in the same left coset of $\mathrm{Stab}_G(x)$ in G. So the number of distinct elements of $G \cdot x$ is the number of distinct left cosets of $\mathrm{Stab}_G(x)$ in G.

Special Case: the conjugation action

Every group G acts on the set of its own elements via the conjugation action, defined for $g, x \in G$ by

$$g \cdot x = gxg^{-1}.$$

Then

- $id_G \cdot x = x$ for all $x \in G$.
- $g \cdot (h \cdot x) = g \cdot (hxh^{-1}) = ghxh^{-1}g^{-1} = ghx(gh)^{-1} = gh \cdot x$.

Under this action, the orbit of $x \in G$ is the conjugacy class of x. The stabilizer of x is the centralizer of x in G. So the Orbit-Stabilizer Theorem tells us that the number of distinct conjugates of x in G is $[G : C_G(x)]$. We already knew this (Theorem 2.2.10).