Lecture 13: Symmetric Groups

The group consisting of all permutations of a set of n elements, under the composition operation, is called the *symmetric group of degree n* and denoted S_n. The order of S_n is $n!$, and it is conventional to label the n objects being permuted as $1, 2, \ldots, n$.

An element of S_4 is a permutation of the set $\{1, 2, 3, 4\}$; this means a function from that set to itself that sends each element to a different image, and hence shuffles the four elements. In S_4, a basic way to represent the permutation
$1 \to 1,\ 2 \to 4,\ 3 \to 2,\ 4 \to 3$ is by the array

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

Disjoint cycle description

Look at the following permutation in S_{14}.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 9 & 8 & 2 & 5 & 1 & 12 & 14 & 6 & 7 & 3 & 13 & 10 & 4 \end{pmatrix}$$

Start with the element 1 and look at what happens to it when you repeatedly apply π.

Disjoint cycle representation

In that there were nine distinct elements in the sequence that started at 1. So the permutation π produces the following *cycle*:

$$1 \to 11 \to 3 \to 8 \to 14 \to 4 \to 2 \to 9 \to 6 \to 1$$

This cycle is often written using the following notation:

$$(1 \;\; 11 \;\; 3 \;\; 8 \;\; 14 \;\; 4 \;\; 2 \;\; 9 \;\; 6).$$

The set $\{1, 2, 3, 4, 6, 8, 9, 11, 14\}$ is called the orbit of 1 under π. We find two more cycles and two more orbits:

- (5), a fixed point
- $(7 \;\; 12 \;\; 13 \;\; 10)$, a cycle of length 4.

We write π as follows as a product of disjoint cycles:

$$\pi = (1 \;\; 11 \;\; 3 \;\; 8 \;\; 14 \;\; 4 \;\; 2 \;\; 9 \;\; 6)(7 \;\; 12 \;\; 13 \;\; 10).$$

Remarks on disjoint cycle representation

- Every permutation can be written as a product (composition) of disjoint cycles. The convention is to not include fixed points (cycles of length 1) in the written description, but you can if you like.
- Disjoint cycles are permutations in the relevant S_n. They commute with each other, because they shuffle separate sets of elements (that's what disjoint means).
- The written description of a permutation as a product of disjoint cycles is unique, except that
  - the order in which the different cycles are written can vary;
  - for each cycle, what matters is the cyclic order, not which element comes first or last, so the expressions $(1\ \ 2\ \ 3\ \ 4)$ and $(3\ \ 4\ \ 1\ \ 2)$ for example represent the same cycle.

Order of an element in a symmetric group

Let G be a group and let $g \in G$. Recall that the order of the element g is the number of elements in the cyclic subgroup generated by g. If this is finite, it is the least k for which $g^k = \mathrm{id}_G$.

Question In S_8, what is the order of the element

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 7 & 8 & 2 & 6 & 3 & 1 \end{pmatrix}?$$

Write $\pi = (1\ \ 4\ \ 8)(2\ \ 5)(3\ \ 7)$. Then $\pi^k = \mathrm{id}$ provided that k is a multiple of both 2 and 3.
The least such k is 6, this is the order of π in S_8.
The order of a permutation is the least common multiple of the cycle lengths in its description as a product of disjoint cycles.

Lecture 14: Conjugacy classes in S_n

In S_7, calculate $\sigma\pi\sigma^{-1}$ (as a product of cycles), where

$$\pi = (1\ 2\ 3\ 4\ 5), \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 3 & 5 & 1 & 4 \end{pmatrix}$$

- The elements 1 and 4 are sent by σ^{-1} to 6 and 7, which are not moved by π, and then mapped respectively back to 1 and 4 by σ. So 1 and 4, which are the images under σ of the fixed points of π, are fixed points of $\sigma\pi\sigma^{-1}$.
- The elements σ^{-1} sends $2, 7, 6, 3, 5$ to $1, 2, 3, 4, 5$ respectively. Then π cycles these around, sending the list $1, 2, 3, 4, 5$ to $2, 3, 4, 5, 1$. Then σ maps the list $2, 3, 4, 5, 1$ back to $7, 6, 3, 5, 2$.
- Overall $\sigma\pi\sigma^{-1} = (2\ 7\ 6\ 3\ 5)$. In particular,$\sigma\pi\sigma^{-1}$ has the same cycle type as π, and the elements that it cycles are the images under σ of those cycled by π.

Conjugates of a general permutation

Suppose $\pi \in S_n$ and write $\pi = \pi_1 \pi_2, \ldots, \pi_k$ be the disjoint cycle representation of π. Let $\sigma \in S_n$. Then

$$\sigma\pi\sigma^{-1} = (\sigma\pi_1\sigma^{-1})(\sigma\pi_2\sigma^{-1})\ldots(\sigma\pi_k\sigma^{-1}).$$

This the disjoint cycle description of $\sigma\pi\sigma^{-1}$, since $(\sigma\pi_i\sigma^{-1})$ cycles the images under σ of the elements cycled by π_i.

Example In S_{10}, write

$$\pi = (1\ 4\ 8\ 7)(2\ 10\ 3\ 6\ 5),\ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 5 & 6 & 3 & 10 & 1 & 4 & 8 & 2 & 7 \end{pmatrix}$$

Then

$$(\sigma\pi\sigma^{-1}) = (9\ \ 3\ \ 8\ \ 4)(5\ \ 7\ \ 6\ \ 1\ \ 10).$$

In particular, elements of S_n that are conjugate to each other have the same cycle type, the same numbers of cycles of each length in their disjoint cycle description.

Same cycle type

Finally, if two elements of S_n do have the same cycle type, they are conjugate in S_n.
In S_8, write

$$\tau = (2\ 5\ 8)(1\ 6\ 7), \quad \pi = (4\ 1\ 6)(5\ 3\ 2).$$

Then τ and π have the same cycle type, and $\sigma\tau\sigma^{-1} = \pi$, provided that the permutation σ maps elements permuted by the 3-cycles of τ to elements permuted by the 3-cycles of π, preserving the cyclic order. For example we could take

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 7 & 8 & 1 & 3 & 2 & 6 \end{pmatrix}.$$

Partitions and Conjugacy Classes

So the conjugacy classes in S_n correspond to the possible cycle types of a permutation in S_n. The number of these is the number of ways to write n as a sum of positive integers: the partitions of n.

If $n = 7$:

- The partition $2 + 2 + 3$ corresponds to permutations with the same cycle type as
$$(1\ 2)(3\ 4)(5\ 6\ 7).$$
The number of these is $\binom{7}{3} \times 2! \times 3 = 210$.
- The partition $1 + 1 + 2 + 3$ corresponds to the permutations with the same cycle type as
$$(1\ 2)(3\ 4\ 5).$$
The number of these is $\binom{7}{3} \times 2! \times \binom{4}{2} = 420$.