Chapter 1

What is a group?

1.1 Examples

This section contains a list of *algebraic structures* with different properties. Although these objects look different from each other, they do have some features in common, for example they are all equipped with algebraic operations (like addition, multiplication etc.). The properties of these operations can be studied and compared. An important theme of group theory (and all areas of abstract algebra) is the distinction between *essential* and *superficial* similarities and differences in algebraic structures.

1. $(\mathbb{Z}, +)$
   $\mathbb{Z}$ is the set of integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
   The "$+$" indicates that we are thinking of $\mathbb{Z}$ as being equipped with addition. This means that given any pair of integers a and b we can produce a new integer by taking their sum $a + b$.

2. $(\mathbb{C}^\times, \times)$
   Here $\mathbb{C}^\times$ denotes the set of *non-zero* complex numbers, and "$\times$" denotes multiplication of complex numbers. So for example
   $$(2 + 3i) \times (1 - i) = 5 + i.$$
   The product of two elements of $\mathbb{C}^\times$ is always an element of $\mathbb{C}^\times$ (we say that $C^\times$ is *closed* under multiplication of complex numbers). So "$\times$" is a *binary operation* on $\mathbb{C}^\times$.

3. $(\mathrm{GL}(2, \mathbb{Q}), \times)$
   Read this as "the general linear group of 2 by 2 matrices over the rational numbers" ("GL" stands for "general linear").
   $$\mathrm{GL}(2, \mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Q}; ad - bc \neq 0 \right\},$$
   so we are talking about the set of 2 by 2 matrices that have rational entries and have non-zero determinant or equivalently that have inverses. The "$\times$" here stands for matrix multiplication. Note that if A and B are elements of $\mathrm{GL}(2, \mathbb{Q})$, then so also are their matrix products AB and BA (which might be not be the same).
   Question: Is this obvious? Why is it true?

4. $(\{1, i, -i, -1\}, \times)$
   Here we are talking about the set of complex fourth roots of unity, under multiplication of complex numbers. Note that this set is closed under multiplication, meaning that the product of any two elements of the set is again in the set. You can check this directly by writing out the whole multiplication table (a worthwhile exercise at this point).

5. Let S_4 denote the set of all permutations of the set $\{a, b, c, d\}$.
Recall that a *permutation* of the set $\{a, b, c, d\}$ is a bijective function from the set to itself. The permutation

$$\begin{array}{ccc} a & \longrightarrow & d \\ b & \longrightarrow & b \\ c & \longrightarrow & a \\ d & \longrightarrow & c \end{array}$$

is sometimes written as $\begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}$.
Given two permutations σ and τ of $\{a, b, c, d\}$, we can *compose* them to form the functions $\sigma \circ \tau$ (σ after τ) and $\tau \circ \sigma$ (τ after σ). This composition works as for any functions and is often referred to as *multiplication* of permutations.

Claim: The functions $\sigma \circ \tau$ and $\tau \circ \sigma$ are again *permutations* of $\{a, b, c, d\}$.
Why is this true? Can you prove it as an exercise?

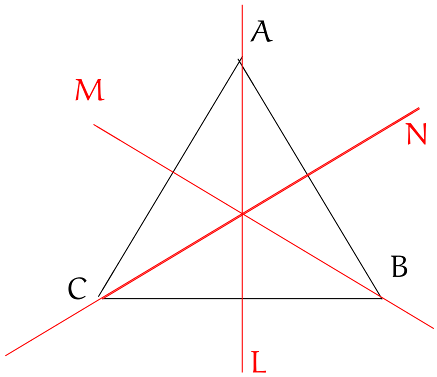
Question: Would you expect $\sigma \circ \tau$ and $\tau \circ \sigma$ to be the same function? If in doubt, try some examples.

Remark: To study group theory and abstract algebra, you may need to relax and expand your understanding of the meaning of the word *multiplication*. Multiplication of integers means something very specific: 5×7 is the number that you get from the addition $5 + 5 + 5 + 5 + 5 + 5 + 5$ or $7 + 7 + 7 + 7 + 7$ (why are these the same?). Mutiplication of real numbers (or complex numbers) are natural extensions of that. In advanced algebra the word "multplication" is often used for operations that don't resemble these familiar ones at all (this already happens in the case of matrix multiplication). It is a good idea to get used to thinking of the work multiplication as just meaning "a way of combining pairs of elements".

6. **General groups of symmetries**
Suppose that P is some connected object in the two-dimensional plane, like a polygon or a line segment or a curve or a disc (*connected* means all in one piece). The following is an informal (and temporary) description of what is meant by a *symmetry* of P. Imagine that P is an object made of a rigid material. If you can pick up this piece of material from the plane and move it around (in 3-dimensional space) without breaking, compressing, stretching or deforming it in any way, and put it back so that the object occupies the same space that it originally did, you have implemented a symmetry of P.

For example, if P is a circular disc, then symmetries of P include rotations about the centre through any angle, reflections in any diameter, and any composition of operations of these kinds. Two symmetries are considered to be the same if P ends up in exactly the same position after both of them - for example in the case of the circular disc, a counter-clockwise rotation about the centre through a full $360°$ is the same as the rotation through $0°$ or the rotation through $720°$.

7. **Symmetries of an equilateral triangle**
Consider an equilateral triangle with vertices labelled A, B, C as in the diagram. For this example it does not matter whether you think of the triangle as consisting just of the vertices and edges or as a solid triangular disc.



The triangle has six symmetries:

- the identity symmetry I, which leaves everything where it is
- the counterclockwise rotation R_{120} through $120°$ about the centroid
- the counterclockwise rotation R_{240} through $240°$ about the centroid
- the reflections in the three medians: call these T_L, T_M, T_N.

Let D_6 denote the set of these six symmetries.
Note that the first three (the rotations) preserve the order in which the vertices A, B, C are encountered as you travel around the perimeter in a counter-clockwise direction; the last three (the reflections) change this order. If you think of the object as a "filled-in" disc, the reflections involve flipping it over and the rotations don't. (Note that the identity permutation is considered to be a rotation, through $0°$ - or any integer multiple of $360°$. It is certainly not a reflection).

Now that we have these six symmetries, we can compose pairs of them together.

Example: We define $R_{120} \circ T_L$ (read the "$\circ$" as "after") to be the symmetry that first reflects the triangle in the vertical line L and then applies the counter-clockwise rotation through $120°$. The overall effect of this leaves vertex B fixed and interchanges the other two, so it is the same as T_M - convince yourself of this, using a physical triangle if necessary. For every pair of our six symmetries, we can figure out what their composition is and write out the whole composition table, which is partly completed below. The entry in this table in the position whose row is labelled with the symmetry τ and whose column is labelled with the symmetry σ is $\tau \circ \sigma$.

$(D_6, \circ)$	I	R_{120}	R_{240}	T_L	T_M	T_N
I	I	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	I	T_M	T_N	T_L
R_{240}						
T_L	T_L	T_N	T_M	I	R_{240}	R_{120}
T_M						
T_N						

Important Exercise: By thinking about the compositions of all these symmetries, verify the part of the above table that is filled in and fill in the rest of it. You should find that each element of D_6 appears exactly once in each row and in each column.

One way to think about symmetries of the triangle is as geometric operations as above. Another is as permutations of the vertices. For example the reflection in the line L fixes the vertex A and swaps the other two, it corresponds to the permutation

$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}.$$

The rotation R_{120} moves vertex A to the position of C, B to the position of A, and C to the position of B. It corresponds to the permutation

$$\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}.$$

Another Important Exercise: Write down the permutations corresponding to the remaining elements of D_6 *and verify that with this interpretation the composition of symmetries as defined above and the multiplication of permutations really amount to the same thing in this context (this means confirming that the permutation corresponding to the composition of two symmetries of the triangle is what you would expect based on the product of the two corresponding permutations).*

Does *every* permutation of the vertices of the triangle arise from a symmetry? If so, what the second important exercise is really saying is that the set of symmetries of an equilateral triangle (with composition) is essentially the same object as the set of permutations of the set $\{A, B, C\}$, with permutation multiplication.

Part of our work in this course will be to precisely formulate what is meant by "essentially the same" here and to develop the conceptual tools and language to discuss situations like this. The examples in this section will hopefully be useful as our account of the subject becomes more technical and abstract.

8. Symmetries of a square
   Consider a square with vertices labelled A, B, C, D (in cyclic order as you travel around the perimeter). Let D_8 denote the set of symmetries of the square.

   Exercise: How many elements does D_8 *have? Describe them in terms of rotations and reflections. Write down the permutation of* $\{A, B, C, D\}$ *corresponding to each one. Does every permutation of this set arise from a symmetry of the square?*