Lecture 3: Group of permutations and symmetries

A permutation of a finite set T is a *bijection* from T to T. This means that every element has a different image, and the image of the function is the whole set T. The permutations of a set form a group under composition.

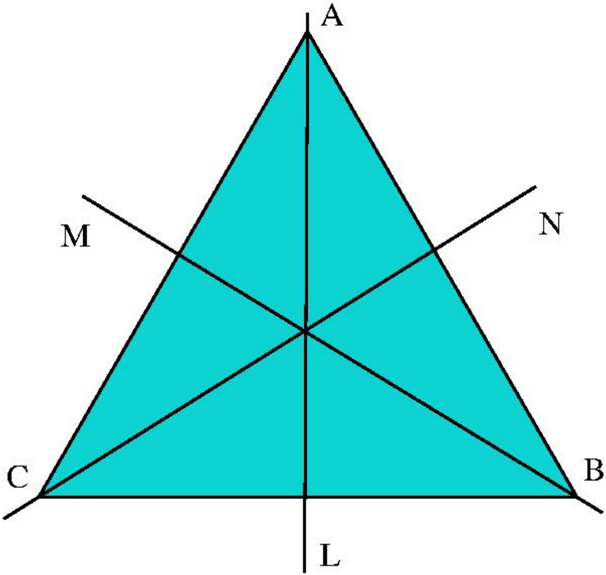
The group of all permutations of a set of n elements is called the symmetric group of degree n and denoted S_n.

How the composition operation works: in the example of S_4.

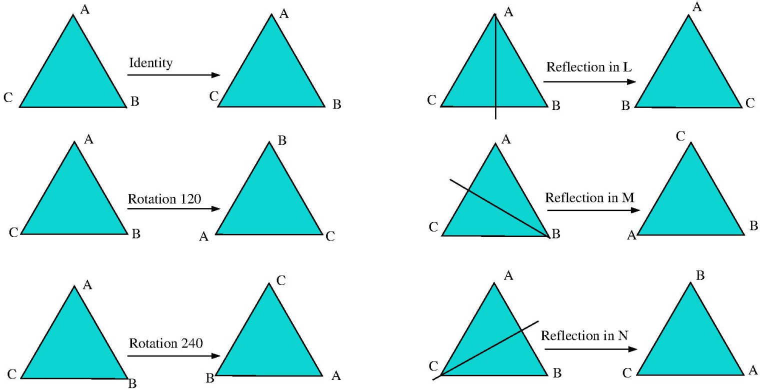
Calculate $\sigma\tau$ and $\tau\sigma$ for

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}.$$

The group of symmetries of the equilateral triangle



Symmetries of the triangle

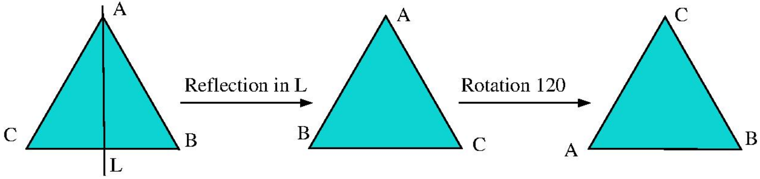


The Group Operation in D_6

The group D_6 of symmetries of the triangle has six elements.

$$D_6 = \{\mathrm{id}, R_{120}, R_{240}, T_L, T_M, T_N\}.$$

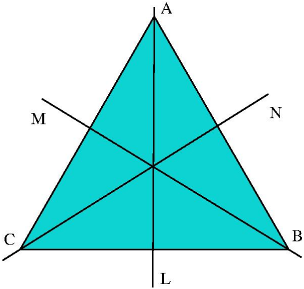
The group operation is composition, denoted by the symbol $\circ$. $R_{120} \circ T_L$ means "R_{120} *after* T_L, the symmetry obtained by applying T_L first and then R_{120}. We can figure out which one it is by watching what happens to the vertices in this composition of symmetries.



Comparing the final position to the starting position, we see that

$$R_{120} \circ T_L = T_M.$$

Group table for D_6



$\circ$	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

In general, the group of symmetries of the regular n-gon is denoted D_{2n} and called the dihedral group of order $2n$. It has $2n$ elements, n rotations (including the identity) and n reflections.

Symmetries of higher-dimensional objects

Like a polygon, a 3-dimensional object has a group of symmetries, which includes rotations and reflections. Think about giving a description of the rotational symmetries of the cube (and the reflections). How many are there? What are the axes about which rotational symmetries occur, and what the the angles of rotation?