Chapter 4

Orthogonality, Inner Products and Projections

4.1 Inner Product Spaces

4.1.1 The ordinary scalar product on $\mathbb{R}^2$

In R^2, the scalar (or dot) product of the vectors $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$ is given by

$$x \cdot y = x_1 y_1 + x_2 y_2 = x^T y = y^T x = y \cdot x.$$

We can interpret the *length* $||x||$ of the vector x as the length of the directed line segment from the origin to (x_1, x_2), which by the Theorem of Pythagoras is $\sqrt{x_1^2 + x_2^2}$ or $\sqrt{x \cdot x}$. Once we have a concept of *length* of a vector, we can define the *distance* $d(x, y)$ between two vectors x and y as the length of ther difference: $d(x, y) = ||x - y||$.

Similarly, from the Cosine Rule we can observe that $x \cdot y = ||x||\,||y|| \cos\theta$, where θ is the angle between the directed line segments representing x and y. In particular, x is orthogonal to y (or $x \perp y$) if and only if $x \cdot y = 0$.

So the scalar product encodes much of the geometry of $\mathbb{R}^2$, and it also provides a mechanism for defining concepts of length, distance and orthogonality on real vector spaces that do not necessarily have an obvious geometric structure.

4.1.2 Real Inner Products

Let V be a vector space over $\mathbb{R}$. An *inner product* on V is a function from $V \times V$ to $\mathbb{R}$ that assigns an element of $\mathbb{R}$ to every order pair of elements of V, and has the following properties. We write $\langle x, y\rangle$ for the inner product of x and y, and write the function as $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

1. Symmetry: $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$
2. Linearity in both slots (bilinearity): For all $x, y, z \in V$ and all $a, b \in \mathbb{R}$, we have $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ and $\langle x, ay + bz\rangle = a\langle x, y\rangle + b\langle x, z\rangle$.
3. Non-negativity: $\langle x, x\rangle \geqslant 0$ for all $x \in V$, and $\langle x, x\rangle = 0$ only if $x = 0_V$.

EXAMPLES We can check that each of the following satisfies the requirements to be an inner product.

1. The ordinary scalar product on $\mathbb{R}^n$.
2. Let C be the vector space of all continuous real-valued functions on the interval $[0, 1]$. The analogue of the ordinary scalar product on C is the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx, \text{ for } f, g \in C.$$

On the space $M_{m\times n}(\mathbb{R})$, the *Frobenius inner product* or *trace inner product* is defined by $\langle A, B\rangle = \text{trace}(A^T B)$. Note that trace ATB is the sum over all positions (i, j) of the products $A_{ij}B_{ij}$. So this is closely related to the ordinary scalar product, if the matrices A and B were regarded as vectors with mn entries over $\mathbb{R}$.

It is possible for a single vector space to have many different inner products defined on it, and if there is any risk of ambiguity we need to specify which one we are considering.