Theorem 3.2.6. *Rank-Nullity Theorem, matrix version Let A be any $m \times n$ matrix, with entries in a field $\mathbb{F}$. Then n is the sum of the dimension of the right nullspace of A and the dimension of the column space of A.*

The dimension of the columns space of a matrix A is called the *column rank* of A.

LEARNING OUTCOMES FOR THIS SECTION

1. *To recall the defintion of a linear transformation*
   as a function between vector spaces that respects the addition and scalar multiplication operations.
2. *To note that left multiplication by any $m \times n$ matrix is a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$,*
   and that the columns of the matrix are the images of the standard basis vectors of $\mathbb{F}^n$
3. *That every linear transformation can be represented as left multiplication by a matrix,*
   after choosing bases for the domain and target spaces. For relatively small and manageable examples, you should be able to write down the matrix that does this, and realize that it depends on the choice of basis (we will come back to this point).
4. *To recognize the terms kernel, image, nullspace, nullity, rank and column space.*
5. *To be able to state and interpret the Rank-Nullity Theorem, in its versions for matrices and for linear transformations*
   The proof is important too, but understanding the statement is more important. One way to think of it informally is that if we apply a linear transformation to a space of dimension n, the image need not have the full dimension n, because some of the elements might be mapped to zero, and so not be "recoverable" in the image (these are the elements of the kernel). But the full dimension n has to be accounted for by the combination of the kernel or the image - their dimensions must add up to n.

3.3 Similarity

In this section we will consider the algebraic relationship between two square matrices that represent the same linear transformation, from a vector space to itself, with respect to different bases.

Example 3.3.1. *Let $T : R^3 \to \mathbb{R}^3$ be the linear transformation defined by $v \to Av$, where*

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix}$$

Let $\mathcal{B}$ be the (ordered) basis of $\mathbb{R}^3$ with elements $b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

What is the matrix A' of T with respect to $\mathcal{B}$?

The columns of A' have the $\mathcal{B}$-coordinates of $T(b_1), T(b_2)$ and $T(b_3)$.

$$T(b_1) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = 2b_1 \Longrightarrow [T(b_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -3b_2 \Longrightarrow [T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$T(b_3) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 14 \end{bmatrix} = 7b_3 \Longrightarrow [T(b_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

We conclude that the matrix of T with respect to $\mathcal{B}'$ is

$$A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

This means: for any $v \in \mathbb{R}^3$,

$$[T(v)]_{\mathcal{B}} = A'[v]_{\mathcal{B}}.$$

The relation of similarity. Staying with this example for now, we consider the relationship between A and A' from another viewpoint. Let P be the matrix with the basis vectors from $\mathcal{B}$ as columns. From Section 3.1, we know that P^{-1} is the change of basis matrix from the standard basis to $\mathcal{B}$. This means that for any element v of $\mathbb{R}^3$, its $\mathcal{B}$-coordinate are given by the matrix-vector product

$$[v]_{\mathcal{B}} = P^{-1}v.$$

Equivalently, if we start with the $\mathcal{B}$-coordinates, then the standard coordinates of v are given by

$$v = P[v]_{\mathcal{B}}.$$

So P itself is the change of basis matrix from $\mathcal{B}$ to the standard basis. Suppose we only knew about A (and had not already calculated A'). We have a basis $\mathcal{B}$ whose columns form the matrix P. To figure out the matrix of T with respect to $\mathcal{B}$:

1. Start with an element of $\mathbb{R}^3$, written in its $\mathcal{B}$-coordinates: $[v]_{\mathcal{B}}$
2. Convert the vector to its standard coordinates (so that we can apply T by multiplying by A): this means taking the product $P[v]_{\mathcal{B}}$
3. Now apply T: this means taking the product $AP[v]_{\mathcal{B}}$. This vector has the standard coordinates of $T(v)$.
4. To convert this to $\mathcal{B}$-coordinates, apply the change of basis matrix from standard to $\mathcal{B}$, which is P^{-1}: this means taking the product $P^{-1}AP[v]_{\mathcal{B}}$. This vector has the $\mathcal{B}$-coordinates of $T(v)$.
5. Conclusion: For any element v of $\mathbb{R}^3$, the $\mathcal{B}$-coordinates of $T(v)$ are given by

$$(P^{-1}AP)[v]_{\mathcal{B}}.$$

This conclusion is saying that the matrix of T with respect to $\mathcal{B}$ is $P^{-1}AP$, where A is the matrix of T with respect to the standard basis, and P is the matrix with the (standard) elements of $\mathcal{B}$ as columns.

Definition 3.3.2. *Let $\mathbb{F}$ be a field. Two matrices A and B in $M_n(\mathbb{F})$ are* similar *if there exists an invertible matrix $P \in M_n(\mathbb{F})$ for which $B = P^{-1}AP$.*

Notes

1. Two distinct matrices in $M_n(\mathbb{F})$ are similar if and only if they represent the same linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^n$, with respect to different bases.
2. As the examples A and A' above show, it is not generally easy to tell by glancing at a pair of square matrices whether they are similar or not, but there is one feature that is easy to check. The *trace* of a square matrix is the sum of the entries on the main diagonal, from top left to bottom right. If two matrices are similar, they have the same trace.
3. Similar matrices also have some other features in common, including having the same determinant. But we have not discussed determinants yet (coming soon).

Item 2. above is a consequence of the following lemma.

Lemma 3.3.3. *Let $A, B \in M_n(\mathbb{F})$. Then* $\mathrm{trace}(AB) = \mathrm{trace}(BA)$.

Consequence: For any square matrix A and any invertible matrix P, both in $M_n(\mathbb{F})$, $\text{trace}(P^{-1}AP) = \text{trace}(AP)P^{-1} = \text{trace}\, A$, so similar matrices always have the same trace.

Proof. (of the Lemma). We calculate the trace of AB in terms of the entries of A and B.

$$\begin{aligned}\text{trace}(AB) &= \sum_{i=1}^{n}(AB)_{ii} \\ &= \sum_{i=1}^{n}\left(sum_{k=1}^{n}A_{ik}B_{ki}\right).\end{aligned}$$

This is the sum over all positions (i,k) of a $n \times n$ matrix of the expressions

(entry in (i,k)-position of A)$\times$ (entry in (k,i)-position of B).

This sum does not change if the roles of A and B are switched, so AB and BA have the same trace. $\square$

In Example 3.3.1, we found that the 3×3 matrix A is similar to the diagonal matrix $A' = \text{diag}(2.-3,7)$. We say that A is *diagonalizable*, which means that it is similar to a diagonal matrix. It $T : \mathbb{R}^3 \to \mathbb{R}^3$ is left multiplication by A, then A' is the matrix of T with respect to the basis $\mathcal{B} = (b_1, b_2, b_3)$, and the basis elements b_1, b_2, b_3 are the columns of the matrix P for which $P^{-1}AP = A'$.
Two (equivalent) observations about this setup:

1. From the diagonal form of A' we have $T(b_1) = 2b_1$, $T(b_2) = -3b_2$ and $T(b_3) = 7(b_3)$. This means that each of the basis elements b_1, b_2, b_3 is mapped by T to a scalar multiple of itself - each of them is an *eigenvector* of T.

2. We can rearrange the version $P^{-1}AP = A'$ to $AP = PA'$. Bearing in mind that $P = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$ and that $A' = diag(2,-3,7)$, this is saying that

$$A\begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \implies \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ 2b_1 & -3b_2 & 7b_3 \\ | & | & | \end{bmatrix}$$

   This means that $Ab_1 = 2b_1$, $Ab_2 = -3b_2$ and $Ab_3 = 7b_3$, so that $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$ consisting of *eigenvectors* of A.

Definition 3.3.4. *Let* $T : V \to V$ *be a linear transformation from a vector space* V *to itself. An* eigenvector *of* T *is a non-zero element* v *of* V *for which* $T(v) = \lambda v$ *for some scalar* λ *(called the* eigenvalue *of* T *to which* v *corresponds).*

In this situation, T can be represented by a diagonal matrix if and only if V has a basis consisting of eigenvectors of T.

Definition 3.3.5. *(Matrix Version). Let* $A \in M_n(\mathbb{F})$*. An* eigenvector *of* A *is a non-zero vector* $v \in \mathbb{F}^n$ *for which* $Av = \lambda v$ *for a scalar* λ *(called the* eigenvalue *of* A *to which* v *corresponds).*

The matrix A is diagonalizable (similar to a diagonal matrix) if and only if there is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.