1. The *cofactor expansion* method is not generally the most efficient way to compute a determinant (it is ok in the 3×3 case). But it can be taken as the *definition* of a determinant.
2. In some special cases, the determinant is easier to compute. If A is upper or lower triangular, then $\det(A)$ is the product of the entries on the main diagonal of A. If A has a square $k \times k$ block A_1 in the upper left, a square $(n-k) \times (n-k)$ block in the lower right, and only zeros in the lower left $(n-k) \times k$ region, then $\det(A) = \det(A_1)\det(A_2)$.
3. For a pair of $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$. This is the *multplicative property* of the determinant, or the Cauchy-Binet formula. It is not obvious at all.
4. A consequence of the multiplicative property of the determinant is that similar matrices have the same determinant and the same characteristic polynomial.

The characteristic polynomial of a 3×3 matrix

Example Using cofactor expansion by the first column, we find that the characteristic polynomial of $B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$ is

$$\begin{aligned}
\det(\lambda I_3 - B) &= \det \begin{bmatrix} \lambda - 5 & -6 & -2 \\ 0 & \lambda + 1 & 8 \\ -1 & 0 & \lambda + 2 \end{bmatrix} \\
&= (\lambda - 5)\left((\lambda + 1)(\lambda + 2) - 0(8)\right) + (-1)\left((-6)(8) - (\lambda + \right. \\
&= (\lambda - 5)(\lambda^2 + 3\lambda + 2) - (2\lambda - 46) \\
&= \lambda^3 - 2\lambda^2 - 15\lambda + 36 \\
&= (\lambda - 3)(\lambda^2 + \lambda - 12) \\
&= (\lambda - 3)(\lambda + 4)(\lambda - 3) \\
&= (\lambda - 3)^2(\lambda + 4)
\end{aligned}$$

Algebraic and Geometric Multiplicity

The eigenvalues of B are 3 (occurring twice as a root of the characteristic polynomial), and -4 (occurring once). We say that 3 has algebraic multplicity 2 and -4 has algebraic multiplicity 1 as an eigenvalue of B. The *geometric multplicity* of each eigenvalue is the dimension of its corresponding eigenspace.

$$3I_3 - B = \begin{bmatrix} -2 & -6 & -2 \\ 0 & 4 & 8 \\ -1 & 0 & 5 \end{bmatrix}$$

The RREF of this matrix is $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and the nullspace consists of all vectors $\begin{bmatrix} 5t \\ -2t \\ t \end{bmatrix}$, where $t \in \mathbb{R}$. This is the eigenspace of B corresponding to $\lambda = 3$. It has dimension 1, so 3 has geometric multiplicity 1 as an eigenvector of B.

Algebraic and Geometric Multiplicity

Theorem The geometric multplicity of an eigenvalue is at most equal to its algebraic multiplicity.

Proof: Suppose that μ has geometric multiplicity k as an eigenvalue of the square matrix $A \in M_n(\mathbb{R})$, and let $\{v_1, \dots, v_k\}$ be a basis for the μ-eigenspace of A. Extend this to a basis $\mathcal{B}$ of $\mathbb{R}^n$, and let P be the matrix whose columns are the elements of $\mathcal{B}$. Then the first k columns of $P^{-1}AP$ have μ in the diagonal position and zeros elsewhere. It follows that $\lambda - \mu$ occurs at least k times as a factor of $\det(\lambda I_n - P^{-1}AP)$.

Corollary A matrix is diagonalizable if and only if the geometric multiplicity of each of its eigenvalues is equal to the algebraic multiplicity.

In R^2, the scalar (or dot) product of the vectors $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$ is given by

$$x \cdot y = x_1 y_1 + x_2 y_2 = x^T y = y^T x = y \cdot x.$$

We can interpret the *length* $||x||$ of the vector x as the length of the directed line segment from the origin to (x_1, x_2), which by the Theorem of Pythagoras is $\sqrt{x_1^2 + x_2^2}$ or $\sqrt{x \cdot x}$. Once we have a concept of *length* of a vector, we can define the *distance* $d(x, y)$ between two vectors x and y as the length of ther difference: $d(x, y) = ||x - y||$.

In R^2, the scalar (or dot) product of the vectors $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$ is given by

$$x \cdot y = x_1 y_1 + x_2 y_2 = x^T y = y^T x = y \cdot x.$$

Similarly, from the Cosine Rule we can observe that $x \cdot y = ||x||\,||y||\cos\theta$, where θ is the angle between the directed line segments representing x and y. In particular, x is orthogonal to y (or $x \perp y$) if and only if $x \cdot y = 0$.

So the scalar product encodes much of the geometry of $\mathbb{R}^2$, and it also provides a mechanism for defining concepts of length, distance and orthogonality on real vector spaces that do not necessarily have an obvious geometric structure.

Real Inner Products

Let V be a vector space over $\mathbb{R}$. An inner product on V is a function from $V \times V$ to $\mathbb{R}$ that assigns an element of $\mathbb{R}$ to every ordered pair of elements of V, and has the following properties. We write $\langle x, y \rangle$ for the inner product of x and y, and write the function as $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

1. Symmetry: $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$
2. Linearity in both slots (bilinearity): For all $x, y, z \in V$ and all $a, b \in \mathbb{R}$, we have $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ and $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$.
3. Non-negativity: $\langle x, x \rangle \geq 0$ for all $x \in V$, and $\langle x, x \rangle = 0$ only if $x = 0_V$.

Examples of inner products

1. The ordinary scalar product on $\mathbb{R}^n$.
2. Let C be the vector space of all continuous real-valued functions on the interval $[0,1]$. The analogue of the ordinary scalar product on C is the inner product given by
$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx, \text{ for } f, g \in C.$$
3. On the space $M_{m\times n}(\mathbb{R})$, the *Frobenius inner product* or *trace inner product* is defined by $\langle A, B\rangle = \text{trace}(A^T B)$. Note that trace$ATB$ is the sum over all positions (i,j) of the products $A_{ij}B_{ij}$. So this is closely related to the ordinary scalar product, if the matrices A and B were regarded as vectors with mn entries over $\mathbb{R}$.

It is possible for a single vector space to have many different inner products defined on it, and if there is any risk of ambiguity we need to specify which one we are considering.