Lecture 19: Section 3.5 The Characteristic Polynomial

The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(\lambda I_2 - A) = (\lambda - a)(\lambda - d) - (-c)(-b) = \lambda^2 - (a+d)\lambda + (ad - bc) = (\lambda - \lambda_1)($$

The sum of the eigenvalues λ_1 and λ_2 is $a + d$, the trace of A.
The product of the eigenvalues λ_1 and λ_2 is $ad - bc$, the determinant of A.

The eigenspace of A corresponding to λ_1 is the nullspace of the matrix $\lambda_1 I_2 - A$. Its non-zero elements are the eigevecors of A corresponding to λ_1.

Section 3.5.1: The Determinant (a digression)

For any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2.$$

- If $ad - bc = 0$, then A is not invertible.
- If $ad - bc \neq 0$, then the equation shows that $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The matrix A has an inverse if and only if $ad - bc \neq 0$. This means that the number $ad - bc$ tells us whether or not the columns of A form a basis of $\mathbb{F}^2$ (or $\mathbb{R}^2$).

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2.$$

The equation above prompts the following definitions

- The number (or field element) $ad - bc$ is the determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denoted by $\det(A)$ or sometimes $|A|$.
- The matrix $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the adjugate (sometimes called the *adjoint*) of A, denoted by $\text{adj}(A)$.

The version of the above equation for a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} ei - fh & -bi + ch & bf - ce \\ -di + fg & ai - cg & -af + cd \\ dh - eg & -ah + bg & ae - bd \end{bmatrix} = (aei - afh - bdi + bfg + cdh - ceg)I_3.$$

The expression $aei - afh - bdi + bfg + cdh - ceg$ is the determinant of A.
The adjugate of A is the matrix on the right.
Its entries are the determinants of the nine 2×2 submatrices of A (some with a sign change).

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix} = (\underbrace{aei - afh - bdi + bfg + cdh - ceg}_{\det(A)})I_3.$$

Definition The minor $M_{i,j}$ of the entry in the (i,j) position of a 3×3 matrix A is the determinant of the 2×2 matrix that remains when Row i and Column j are deleted from A.

Definition The cofactor $C_{i,j}$ of the entry in the (i,j) position of a 3×3 matrix A is either equal to $M_{i,j}$ or to $-M_{i,j}$, according to the following pattern of signs: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Definition The adjugate of the 3×3 matrix A is the matrix that has $C_{j,i}$ in the (i,j)-position. It is the *transpose* of the matrix of cofactors of A.

By looking at any of the three entries on the main diagonal of the product $A \times \text{adj}(A)$, we can give the following description of the determinant of a 3×3 matrix.

Definition The determinant of a 3×3 matrix is A can be found by choosing any row or column of A, multplying each entry of that row or column by its own cofactor, and adding the results.

Determinant Properties

1. Each of the definitions above applies to $n \times n$ matrices in general, and gives us a way to recursively define a $n \times n$ determinant, in terms of $(n-1) \times (n-1)$ determinants.
2. The *cofactor expansion* method is not generally the most efficient way to compute a determinant (it is ok in the 3×3 case). But it can be taken as the *definition* of a determinant.
3. In some special cases, the determinant is easier to compute. If A is upper or lower triangular, then $\det(A)$ is the product of the entries on the main diagonal of A. If A has a square $k \times k$ block A_1 in the upper left, a square $(n-k) \times (n-k)$ block in the lower right, and only zeros in the lower left $(n-k) \times k$ region, then $\det(A) = \det(A_1)\det(A_2)$.
4. For a pair of $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$. This is the *multplicative property* of the determinant, or the Cauchy-Binet formula. It is not obvious at all.

Example Using cofactor expansion by the first column, we find that the characteristic polynomial of $B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$ is

$$\begin{aligned}
\det(\lambda I_3 - B) &= \det \begin{bmatrix} \lambda - 5 & -6 & -2 \\ 0 & \lambda + 1 & 8 \\ -1 & 0 & \lambda + 2 \end{bmatrix} \\
&= (\lambda - 5)\left((\lambda + 1)(\lambda + 2) - 0(8)\right) + (-1)\left((-6)(8) - (\lambda + \right. \\
&= (\lambda - 5)(\lambda^2 + 3\lambda + 2) - (2\lambda - 46) \\
&= \lambda^3 - 2\lambda^2 - 15\lambda + 36 \\
&= (\lambda - 3)(\lambda^2 + \lambda - 12) \\
&= (\lambda - 3)(\lambda + 4)(\lambda - 3) \\
&= (\lambda - 3)^2(\lambda + 4)
\end{aligned}$$

The eigenvalues of B are 3 (occurring twice as a root of the characteristic polynomial), and -4 (occurring once). We say that 3 has algebraic multplicity 2 and -4 has algebraic multiplicity 1 as an eigenvalue of B. The *geometric multplicity* of each eigenvalue is the dimension of its corresponding eigenspace.

$$3I_3 - B = \begin{bmatrix} -2 & -6 & -2 \\ 0 & 4 & 8 \\ -1 & 0 & 5 \end{bmatrix}$$

The RREF of this matrix is $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and the nullspace consists of all vectors $\begin{bmatrix} 5t \\ -2t \\ t \end{bmatrix}$, where $t \in \mathbb{R}$. This is the eigenspace of B corresponding to $\lambda = 3$. It has dimension 1, so 3 has geometric multiplicity 1 as an eigenvector of B.

Algebraic and Geometric Multiplicity

Theorem The geometric multplicity of an eigenvector is at most equal to its algebraic multiplicity.

Corollary A matrix is diagonalizable if and only if the geometric multiplicity of each of its eigenvalues is equal to the algebraic multiplicity.