# Lecture 18: At most $n$ distinct eigenvalues

Theorem If $v_1, \dots, v_k$ are eigenvectors of $A \in M_n(\mathbb{F})$, corresponding to different eigenvalues, then $\{v_1, \dots, v_k\}$ is a linearly independent set.

The following consequence of the Theorem suggests that we may have some chance of being able to find the eigenvalues of a $n \times n$ matrix, or at least that there are not too many of them.

Corollary Let $A \in M_n(\mathbb{F})$. Then $A$ has at most $n$ distinct eigenvalues in $\mathbb{F}$.

Proof If $A$ has $k$ distinct eigenvalues, with corresponding eigenvectors $v_1, \dots, v_k$ in $\mathbb{F}^n$, then $k$ cannot exceed the dimension of $\mathbb{F}^n$, since $\{v_1, \dots, v_k\}$ is a linearly indepedent set in $\mathbb{F}^n$. Hence $k \leq n$.

Corollary If $A \in M_n(\mathbb{F})$ has $n$ distinct eigenvalues in $\mathbb{F}$, then $A$ is diagonalizable in $M_n(\mathbb{F})$.

# A note about (non)-diagonalizability

For $A \in M_n(\mathbb{F})$, it does not always happen that $\mathbb{F}^n$ has a basis consisting of eigenvectors of $A$.

## Examples

1. The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable in $M_2(\mathbb{C})$ but not in $M_2(\mathbb{R})$.
2. The matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable even over $\mathbb{C}$.

# Learning outcomes for Section 3.4

1. To define an eigenvector of a linear transformation or of a square matrix.
2. To know that eigenvectors corresponding to different eigenvalues are linearly independent.
3. And that this means a $n \times n$ matrix can have at most $n$ distinct eigenvalues
4. and that it is diagonalizable if it does have $n$ distinct eigenvalues.

Example Find a matrix $P$ with $P^{-1}AP$ diagonal, where $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

To answer this, we need to find two linearly independent eigenvectors of $A$. These are non-zero solutions of

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{array}{rcl} 2x + 2y & = & \lambda x \\ x + 3y & = & \lambda y \end{array} \implies \begin{array}{rcl} 0 & = & (\lambda - 2)x - 2y \\ 0 & = & -x + (\lambda - 3)y \end{array} \implies \begin{bmatrix} \lambda - 2 & -2 \\ -1 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So we are looking for non-zero solutions $\begin{bmatrix} x \\ y \end{bmatrix}$ of the system

$$\begin{bmatrix} \lambda - 2 & -2 \\ -1 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These can occur only if the coefficient matrix is non-invertible. If it is invertible, the only solution is $x = y = 0$.
A $2 \times 2$ matrix is non-invertible if and only if its determinant is 0.

$$\det \begin{bmatrix} \lambda - 2 & -2 \\ -1 & \lambda - 3 \end{bmatrix} = (\lambda - 2)(\lambda - 3) - (-2)(-1) = \lambda^2 - 5\lambda + 4.$$

# The Characteristic Polynomial of a $2 \times 2$ matrix

The characteristic polynomial of $A$ is

$$\det(\lambda I - A) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1).$$

The eigenvalues of $A$ are the solutions of the characteristic equation $\det(\lambda I - A) = 0$, 1 and 4. The eigenspace of $A$ corresponding to $\lambda = 1$ is the set of all solutions of the system

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{bmatrix} 1-2 & -2 \\ -1 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the nullspace of the matrix $1I - A = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$, which is

$$\left\{ \begin{bmatrix} -2t \\ t \end{bmatrix}, t \in \mathbb{R} \right\}.$$

An eigenvector of $A$ for $\lambda = 1$ is any non-zero element of this space, for example $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.