In last week's example, we found that the 3×3 matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix}$$

is similar to the diagonal matrix $A' = \mathrm{diag}(2, -3, 7)$. We say that A is diagonalizable, which means that it is similar to a diagonal matrix. We had

$$A' = P^{-1}AP,$$

where $P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$.

The columns of P are b_1, b_2, b_3, and A' is the matrix with respect to the basis $\mathcal{B} = \{b_1, b_2, b_3\}$ of the linear transformation of $\mathbb{R}^3$ defined by $v \to Av$.

1. From the diagonal form of A' we have $T(b_1) = 2b_1$, $T(b_2) = -3b_2$ and $T(b_3) = 7b_3$. This means that each of the basis elements b_1, b_2, b_3 is mapped by T to a scalar multiple of itself - each of them is an *eigenvector* of T.
2. We can rearrange the version $P^{-1}AP = A'$ to $AP = PA'$. Bearing in mind that $P = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$ and that $A' = \mathrm{diag}(2, -3, 7)$, this is saying that

$$A \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ 2b_1 & -3b_2 & 7b_3 \\ | & | & | \end{bmatrix}$$

This means that $Ab_1 = 2b_1$, $Ab_2 = -3b_2$ and $Ab_3 = 7b_3$, so that $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$ consisting of *eigenvectors* of A.

Definition Let $T : V \to V$ be a linear transformation from a vector space V to itself. An eigenvector of T is a non-zero element v of V for which $T(v) = \lambda v$ for some scalar λ (called the eigenvalue of T to which v corresponds).

In this situation, T can be represented by a diagonal matrix if and only if V has a basis consisting of eigenvectors of T.

Definition (Matrix Version). Let $A \in M_n(\mathbb{F})$. An eigenvector of A is a non-zero vector $v \in \mathbb{F}^n$ for which $Av = \lambda v$ for a scalar λ (called the eigenvalue of A to which v corresponds).

The matrix A is diagonalizable (similar to a diagonal matrix) if and only if there is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

Working with eigenvectors

Given $A \in M_n(\mathbb{F})$ and $v \in \mathbb{F}^n$, it is straightforward task to determine whether v is an eigenvector of A.

Example Show that $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} -2 & 9 \\ 8 & 4 \end{bmatrix}$ and find the corresponding eigenvalue.

$$\begin{bmatrix} -2 & 9 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 10 \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

The corresponding eigenvalue is 10.

Working with eigenvectors

Given a vector v, we can construct a matrix that has v as an eigenvector, with our favourite scalar as an eigenvalue.

Example Find a matrix $A \in M_3(\mathbb{R})$ that has $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as an eigenvector, corresponding to the eigenvalue 28.

Write u_1, u_2, u_3 as the three rows of A. What we need is that $u_1 v = 28(1) = 28$, $u_2 v = 28(2) = 56$, $u_3 v = 28(3) = 84$. An easy way to arrange this is to choose

$$u_1 = (28 \quad 0 \quad 0), u_2 = (0 \quad 28 \quad 0), u_3 = (0 \quad 0 \quad 28),$$

so that $A = 28I_3$. We can find other answers, for example

$$u_1 = (3 \quad 2 \quad 3), \quad u_2 = (0 \quad -2 \quad 20), \quad u_3 = (5 \quad 2 \quad 25)$$

to get $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & -2 & 20 \\ 5 & 2 & 25 \end{bmatrix}$. It is easily confirmed that $Av = 28v$.

How to find eigenvectors?

A harder problem is to find the eigenvectors of a matrix or linear trasformation, given only the matrix or linear transformation itself. For example, suppose that

$$B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}.$$

Finding an eigenvector of B means finding solutions for x, y, z and λ, to the following system of equations, where the values of x, y, z are not all zero.

$$\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If λ is regarded as a variable, this is not a system of linear equations. Where to begin?
The key is to find the *eigenvalues* first. To see why, we show that the number of distinct eigenvalues of a $n \times n$ matrix cannot exceed n.

Eigenvectors for distinct eigenvalues are independent

Theorem Let $A \in M_n(\mathbb{F})$ and let $v_1, \dots, v_k$ be eigenvectors of A in $\mathbb{F}^n$, corresponding to *distinct* eigenvalues $\lambda_1, \dots, \lambda_k$ of A. Then $\{v_1, \dots, v_k\}$ is a linearly independent subset of $\mathbb{F}^n$.

Idea of Proof: First suppose that $k = 2$, and suppose that $a_1 v_1 + a_2 v_2 = 0$, for scalars a and b in $\mathbb{F}$. We need to show that $a_1 = a_2 = 0$. Multiplying the expression $a_1 v_1 + a_2 v_2$ on the left by A, we have

$$a_1 A v_1 + a_1 A v_2 = 0 \Longrightarrow a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0.$$

Multiplying the same expression by the scalar λ_1 gives

$$a_1 \lambda_1 v_1 + a_2 \lambda_1 v_2 = 0.$$

Subtracting one of these expressions from the other gives

$$a_2(\lambda_1 - \lambda_2) v_2 = 0.$$

At most n distinct eigenvalues

The following consequence of the Theorem suggests that we may have some chance of being able to find the eigenvalues of a $n \times n$ matrix, or at least that there are not too many of them.

Corollary Let $A \in M_n(\mathbb{F})$. Then A has at most n distinct eigenvalues in $\mathbb{F}$.

Proof If A has k distinct eigenvalues, with corresponding eigenvectors $v_1, \dots, v_k$ in $\mathbb{F}^n$, then k cannot exceed the dimension of $\mathbb{F}^n$, since $\{v_1, \dots, v_k\}$ is a linearly indepedent set in $\mathbb{F}^n$. Hence $k \leq n$.