Example Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $v \to Av$, where

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix}$$

Let $\mathcal{B}$ be the (ordered) basis of $\mathbb{R}^3$ with elements

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \ b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

What is the matrix A' of T with respect to $\mathcal{B}$?

The columns of A' have the $\mathcal{B}$-coordinates of $T(b_1)$, $T(b_2)$ and $T(b_3)$.

$\mathcal{B}$-coordinates of $T(b_i)$

$$T(b_1) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = 2b_1 \Longrightarrow [T(b_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -3b_2 \Longrightarrow [T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$T(b_3) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 14 \end{bmatrix} = 7b_3 \Longrightarrow [T(b_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

The matrix of T with respect to $\mathcal{B}'$ is

$$A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

This means: for any $v \in \mathbb{R}^3$,

$$[T(v)]_{\mathcal{B}} = A'[v]_{\mathcal{B}}.$$

Change of basis again

For this example for now, we consider the relationship between A and A' from another viewpoint. Let P be the matrix with the basis vectors from $\mathcal{B}$ as columns. From Section 3.1, we know that P^{-1} is the change of basis matrix from the standard basis to $\mathcal{B}$. This means that for any element v of $\mathbb{R}^3$, its $\mathcal{B}$-coordinate are given by the matrix-vector product

$$[v]_{\mathcal{B}} = P^{-1}v.$$

Equivalently, if we start with the $\mathcal{B}$-coordinates, then the standard coordinates of v are given by

$$v = P[v]_{\mathcal{B}}.$$

So P itself is the change of basis matrix from $\mathcal{B}$ to the standard basis.

Similarity

Suppose we only knew about A (and had not already calculated A'). We have a basis $\mathcal{B}$ whose columns form the matrix P. To figure out the matrix of T with respect to $\mathcal{B}$:

1. Start with an element of $\mathbb{R}^3$, written in its $\mathcal{B}$-coordinates: $[v]_{\mathcal{B}}$
2. Convert the vector to its standard coordinates (so that we can apply T by multiplying by A): this means taking the product $P[v]_{\mathcal{B}}$
3. Now apply T: this means taking the product $AP[v]_{\mathcal{B}}$. This vector has the standard coordinates of $T(v)$.
4. To convert this to $\mathcal{B}$-coordinates, apply the change of basis matrix from standard to $\mathcal{B}$, which is P^{-1}: this means taking the product $P^{-1}AP[v]_{\mathcal{B}}$. This vector has the $\mathcal{B}$-coordinates of $T(v)$.
5. Conclusion: For any element v of $\mathbb{R}^3$, the $\mathcal{B}$-coordinates of $T(v)$ are given by $(P^{-1}AP)[v]_{\mathcal{B}}$.

Definition Let $\mathbb{F}$ be a field. Two matrices A and B in $M_n(\mathbb{F})$ are *similar* if there exists an invertible matrix $P \in M_n(\mathbb{F})$ for which $B = P^{-1}AP$.

Notes

1. Two distinct matrices in $M_n(\mathbb{F})$ are similar if and only if they represent the same linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^n$, with respect to different bases.
2. It is not generally easy to tell by glancing at a pair of square matrices whether they are similar or not, but there is one feature that is easy to check. The trace of a square matrix is the sum of the entries on the main diagonal, from top left to bottom right. If two matrices are similar, they have the same trace.
3. Similar matrices also have some other features in common, such as having the same determinant (more on that later).

Similar matrices have the same trace

Lemma Let $A, B \in M_n(\mathbb{F})$. Then $\text{trace}(AB) = \text{trace}(BA)$.

Consequence: For any square matrix A and any invertible matrix P, both in $M_n(\mathbb{F})$,

$$\text{trace}(P^{-1}AP) = \text{trace}(AP)P^{-1} = \text{trace}A,$$

so similar matrices always have the same trace.

Diagonalizability

In the example, we found that the 3×3 matrix A is similar to the diagonal matrix $A' = \text{diag}(2. -3, 7)$. We say that A is diagonalizable, which means that it is similar to a diagonal matrix.

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is left multiplication by A, then A' is the matrix of T with respect to the basis $\mathcal{B} = (b_1, b_2, b_3)$, and the basis elements b_1, b_2, b_3 are the columns of the matrix P for which $P^{-1}AP = A'$.

Two equivalent interpretations

1. From the diagonal form of A' we have $T(b_1) = 2b_1$, $T(b_2) = -3b_2$ and $T(b_3) = 7b_3$. This means that each of the basis elements b_1, b_2, b_3 is mapped by T to a scalar multiple of itself - each of them is an *eigenvector* of T.
2. We can rearrange the version $P^{-1}AP = A'$ to $AP = PA'$. Bearing in mind that $P = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$ and that $A' = diag(2, -3, 7)$, this is saying that

$$A \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix}$$

This means that $Ab_1 = 2b_1$, $Ab_2 = -3b_2$ and $Ab_3 = 7b_3$, so that $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$ consisting of *eigenvectors* of A.

Eigenvectors

Definition Let $T : V \to V$ be a linear transformation from a vector space V to itself. An eigenvector of T is a non-zero element v of V for which $T(v) = \lambda v$ for some scalar λ (called the eigenvalue of T to which v corresponds).

In this situation, T can be represented by a diagonal matrix if and only if V has a basis consisting of eigenvectors of T.

Definition (Matrix Version). Let $A \in M_n(\mathbb{F})$. An eigenvector of A is a non-zero vector $v \in \mathbb{F}^n$ for which $Av = \lambda v$ for a scalar λ (called the eigenvalue of A to which v corresponds).

The matrix A is diagonalizable (similar to a diagonal matrix) if and only if there is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.