$\Phi : V \to W$ is a linear transformation of finite dimensional vector spaces.

Definition The kernel of Φ , denoted $\ker \Phi$, is the set of elements of V whose image is the zero vector of W.

$$\ker \Phi = \{v \in V : \Phi(v) = 0_W\} \subseteq V.$$

Definition The *image* of Φ, denoted $\operatorname{image} \Phi$, is the subset of W consisting of the images of all the elements of V.

$$\operatorname{image} \phi = \{\phi(v) : v \in V\} \subseteq W.$$

The kernel and image of ϕ are subspaces of V and W.

The nullspace of a matrix

Example For the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined as left multiplication by the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix}$, the *kernel* consists of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the matrix context, this is referred to as the (right) *nullspace* of A. We can find it by row reduction; in this example it consists of all vectors of the form $t \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$ where $t \in \mathbb{R}$ - a subspace of dimension 1 of $\mathbb{R}^3$.

The nullspace of $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \end{bmatrix}$ is the linear span of $\begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$.

The image of the "left multiplication by A" linear transformation is the linear span of the three columns of A. In the matrix context, it is called the column space of A. In this example, it is all of $\mathbb{R}^2$, since the first two columns of A (for example) span $\mathbb{R}^2$.

We can note that in this example, the kernel (nullspace) and image (column space) have dimension 1 and 2, and 1+2=3, and 3 is the dimension of the domain $\mathbb{R}^3$. This is not a coincidence, but a case of the Rank-Nullity Theorem.

The Rank-Nullity Theorem relates the dimensions of the kernel, image and domain of a linear transformation. The dimension of the image of a linear trasformation is called its *rank*, and the dimension of the kernel is called the *nullity*.

Theorem (Rank-Nullity Theorem) Let $\phi : V \to W$ be a linear transformation, where V and W are finite-dimensional vector spaces over a field $\mathbb{F}$. Then

$$\dim(\ker\phi) + \text{rank}\phi = \dim V.$$

Informally, the Rank-Nullity Theorem says that the full dimension of the domain must be accounted for in the combination of the kernel and the image.

Learning Outcomes for Section 3.2

1. *To recall the definition of a linear transformation*
   as a function between vector spaces that respects the addition and scalar multiplication operations.
2. *To note that left multiplication by any $m \times n$ matrix is a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$,*
   and that the columns of the matrix are the images of the standard basis vectors of $\mathbb{F}^n$
3. *That every linear transformation can be represented as left multiplication by a matrix,*
   For relatively small and manageable examples, you should be able to write down the matrix that does this, and realize that it depends on the choice of basis (we will come back to this point).
4. *To recognize the terms kernel, image, nullspace, nullity, rank and column space.*
5. *To be able to state and interpret the Rank-Nullity Theorem, in its versions for matrices and for linear transformations*

Example Let $T : R^3 \to \mathbb{R}^3$ be the linear transformation defined by $v \to Av$, where

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix}$$

Let $\mathcal{B}$ be the (ordered) basis of $\mathbb{R}^3$ with elements

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \; b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \; b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

What is the matrix A' of T with respect to $\mathcal{B}$?

The columns of A' have the $\mathcal{B}$-coordinates of $T(b_1)$, $T(b_2)$ and $T(b_3)$.

$\mathcal{B}$-coordinates of $T(b_i)$

$$T(b_1) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = 2b_1 \implies [T(b_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -3b_2 \implies [T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$T(b_3) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 14 \end{bmatrix} = 7b_3 \implies [T(b_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

The matrix of T with respect to $\mathcal{B}'$ is

$$A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

This means: for any $v \in \mathbb{R}^3$,

$$[T(v)]_{\mathcal{B}} = A'[v]_{\mathcal{B}}.$$

Change of basis again

For this example for now, we consider the relationship between A and A' from another viewpoint. Let P be the matrix with the basis vectors from $\mathcal{B}$ as columns. From Section 3.1, we know that P^{-1} is the change of basis matrix from the standard basis to $\mathcal{B}$. This means that for any element v of $\mathbb{R}^3$, its $\mathcal{B}$-coordinate are given by the matrix-vector product

$$[v]_{\mathcal{B}} = P^{-1}v.$$

Equivalently, if we start with the $\mathcal{B}$-coordinates, then the standard coordinates of v are given by

$$v = P[v]_{\mathcal{B}}.$$

So P itself is the change of basis matrix from $\mathcal{B}$ to the standard basis.

Suppose we only knew about A (and had not already calculated A'). We have a basis $\mathcal{B}$ whose columns form the matrix P. To figure out the matrix of T with respect to $\mathcal{B}$:

1. Start with an element of $\mathbb{R}^3$, written in its $\mathcal{B}$-coordinates: $[v]_{\mathcal{B}}$
2. Convert the vector to its standard coordinates (so that we can apply T by multiplying by A): this means taking the product $P[v]_{\mathcal{B}}$
3. Now apply T: this means taking the product $AP[v]_{\mathcal{B}}$. This vector has the standard coordinates of $T(v)$.
4. To convert this to $\mathcal{B}$-coordinates, apply the change of basis matrix from standard to $\mathcal{B}$, which is P^{-1}: this means taking the product $P^{-1}AP[v]_{\mathcal{B}}$. This vector has the $\mathcal{B}$-coordinates of $T(v)$.
5. Conclusion: For any element v of $\mathbb{R}^3$, the $\mathcal{B}$-coordinates of $T(v)$ are given by $(P^{-1}AP)[v]_{\mathcal{B}}$.