General advice on calculus exam questions

- There will be three calculus questions, one based on each of the three chpaters of our lecture notes.
- Each question will have four parts, (a),(b),(c),(d), all of the same weight (similar to the 2019 summer paper).
- All questions must be answered for full marks.

(a) (5 marks) Determine the cardinality of each of the following finite subsets of $\mathbb{R}$.

(i) $\mathbb{Z} \cap [1, 3]$.

(ii) $\mathbb{Z} \cap (1, 3)$

(iii) The set of integers x for which $x^2 \leq 10$.

(iv) The set of prime numbers that are less than 20.

(v) The set of solutions in $\mathbb{R}$ of the equation $x^4 - 1 = 0$.

1 2 3

$|\mathbb{Z} \cap [1,3]| = 3$

(i) $\boxed{3}$ $\qquad \boxed{\mathbb{Z} \cap [1,3] = \{1,2,3\}}$

(ii) 1 $\qquad \boxed{\mathbb{Z} \cap (1,3) = \{2\}}$ so the cardinality is 1

(iii) 7 $\qquad \boxed{\{-3, -2, -1, 0, 1, 2, 3\}}$

(iv) 8 $\qquad 2, 3, 5, 7, 11, 13, 17, 19$

(v) 2 $\qquad (1 \text{ and } -1)$ $\qquad (x^4 - 1) = (x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x^2+1)$

2019 Calculus Question on Chapter 2 (four parts)

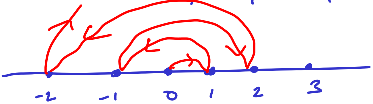
(b) (5 marks) What does it mean for an infinite set to be *countable*? Show that the set $\mathbb{Z}$ of integers is countable, and give an example of an infinite set that is not countable.

An infinite set is countable if its elements can be put in bijective correspondence with the natural numbers.

$\mathbb{Z}$ is countable:
A bijective correspondence between $\mathbb{Z}$ and $\mathbb{N}$ is given by this table:

$\mathbb{N}$	$\mathbb{Z}$
1	0
2	1
3	-1
4	2
5	-2
6	3
$\vdots$	$\vdots$

An example of an uncountable set is the open interval $(0,1)$.



(c) (5 marks) Let

$$S := \left\{ \frac{n^2 - 2n}{n^2 + 1} : n \in \mathbb{Z} \right\}.$$

Determine, with explanation, the infimum and supremum of S. Does S have a maximum and/or a minimum element?

What are the elements of S?

What are the greatest elements and the least elements?

Elements of S are $\frac{n^2-2n}{n^2+1}$ where n is an integer.

We can write an element of S as

$$\frac{n^2+1-2n-1}{n^2+1} = \frac{n^2+1}{n^2+1} - \frac{2n+1}{n^2+1} = 1 - \boxed{\frac{2n+1}{n^2+1}}$$

A few examples of elements of S:

$n=0$: 0 $\quad n=1$: $-\frac{1}{2}$ $\quad n=2$: 0 $\quad n=3$: $\frac{3}{10}$ $\quad n=4$: $\frac{8}{17}$ ...

$n=-2$: $8/17$

$$S = \left\{ 1 - \frac{2n+1}{n^2+1} : n \in \mathbb{Z} \right\}$$

If $n > 0$, the element of S determined by n is < 1

What's the biggest value that we can subtract from 1 in an element of S?

e.g. take $n = 1$: $1 - 3/2 = \boxed{-1/2}$

$n = 2$: $1 - 5/5 = 1 - 1 = 0$

~~no~~ $\frac{2n+1}{n^2+1}$ decreases as n increases from 2

Minimum value is $-1/2$, obtained when $n = 1$.

If $n < 0$, and $n \in \mathbb{Z}$, $2n - 1 < 0$ also and corresponding elements of S are greater than 1

$n = -1$ $\quad 1 - \frac{-1}{2} = 3/2 \in S$

$\boxed{n = -2}$ $\quad 1 - \frac{-3}{5} = \boxed{8/5}$

$n = -3$ $\quad 1 - -5/10 = 3/2$

$n = -4$ $\quad 1 - -7/17$

Highest value is achieved when $n = -2$,

$8/5 \in S$, $\quad \max(S) = 8/5$

Conclusion $\quad \max(S) = \sup(S) = 8/5$

$\min(S) = \inf(S) = -1/2$

(d) (5 marks) Give an example of a set of real numbers that
- (i) has a supremum but no infimum;
- (ii) has a maximum and an infimum but no minimum;
- (iii) has neither a supremum nor an infimum.

(i) The set of negative integers
(any set that's bounded above but not below)

(ii) $(0, 1]$ maximum is 1
infimum is 0
no minimum element

(iii) $\mathbb{R}$ (or $\mathbb{Z}$, or $\mathbb{Q}$)
any set that has neither an upper nor lower bound