Section 3.1: Introduction to sequences and series

Question 51

Does it make sense to talk about the "number"

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots ?$$

sum of the expression $\frac{1}{n^2}$ as n goes from 1 to infinity

Σ : summation sign

- $1 + \frac{1}{4} = 1.25$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.423611$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10)^2} \approx 1.549767$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(200)^2} \approx 1.639947$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10000)^2} \approx 1.644834$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(100000)^2} \approx 1.644924$

$$\frac{\pi^2}{6} \approx 1.644934$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges to the number $\frac{\pi^2}{6}$ (we will have precise definitions for the highlighted terms a bit later).

This fact is remarkable - there is no obvious connection between π and squares of the form $\frac{1}{n^2}$; moreover all the terms in the series are rational but $\frac{\pi^2}{6}$ is certainly not.
This example gives us in principle a way of calculating the digits of π or at least of π^2. (In practice there are similar but better ways, as the convergence in this example is very slow).

Another Example

Example 52

What about

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ?$$

Try experimenting with initial segments again :

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50} \approx 4.4992$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} \approx 5.1874$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1000} \approx 7.4855$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50000} \approx 11.3970$

This is called harmonic series. Recall from Section 1.5 that $\int_1^{\infty} \frac{1}{x}\,dx$ is divergent (area is infinite)

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

Another Example . . .

Example 53

What about

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ...?$$

Experimenting reveals

- $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$
- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} = \frac{341}{1024} \approx 0.33301$
- $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots + \frac{1}{2^{14}} \approx 0.3333$

These calculations can be verified directly using properties of sums of geometric progressions. It appears that this series is converging (quite fast) to $\frac{1}{3}$.

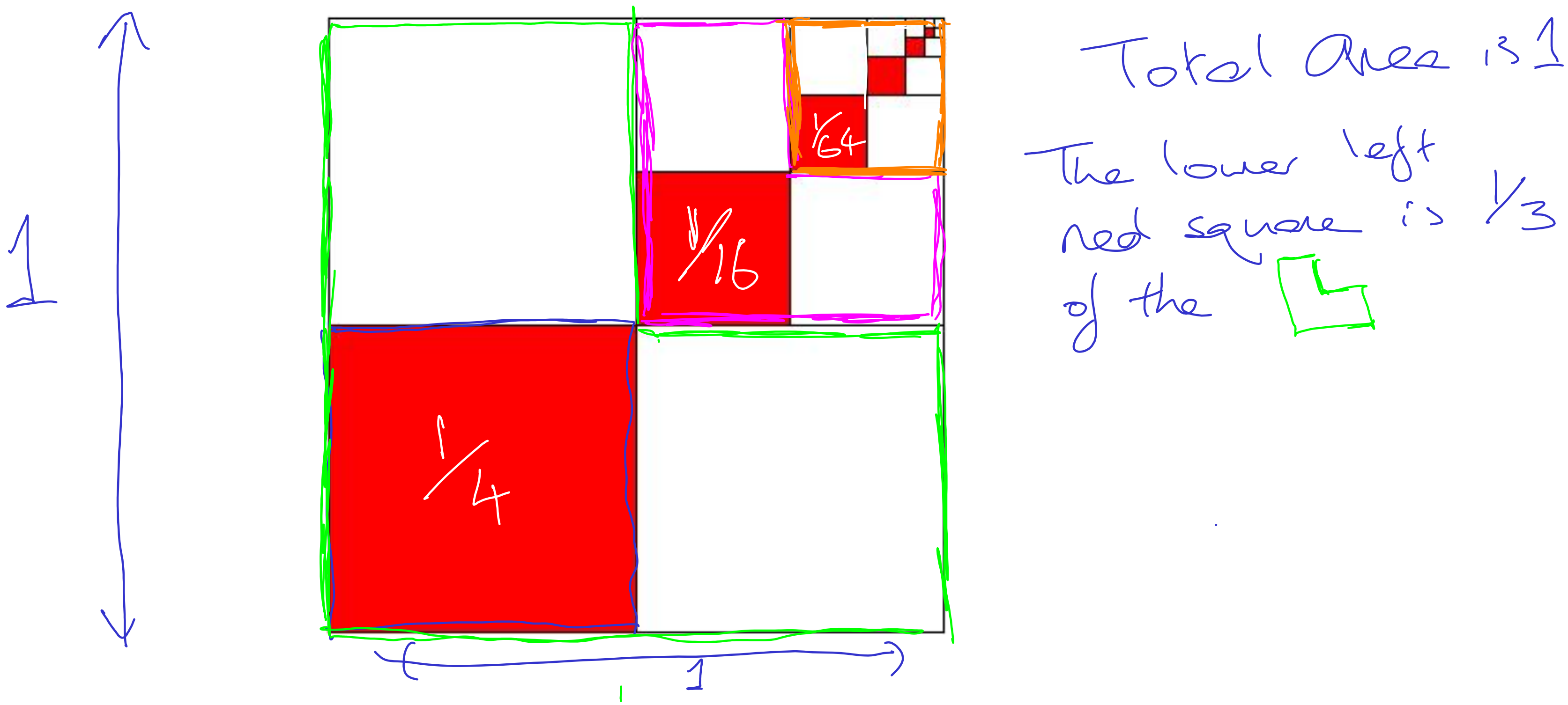
Another Example . . .

Example 54

What about

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots ?$$

The following picture gives some graphical evidence for this hypothesis.



A last example

Example 55

Does it make sense to talk about

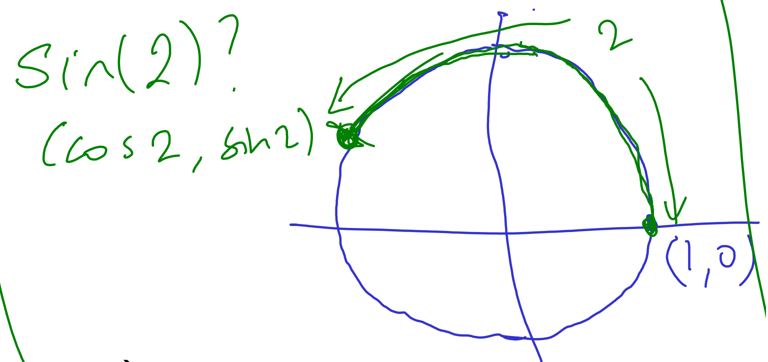
$5! = 5\times4\times3\times2\times1$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

as a function of x?

If it does, then f must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of x must somehow make sense.

- $x = 0$: $f(0) = 0$
- $x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) \approx 0.9999$ (six terms)
- $x = \frac{\pi}{6}$: $f(\frac{\pi}{6}) \approx 0.5000$ (six terms)
- $x = \frac{\pi}{3}$: $f(\frac{\pi}{3}) \approx 0.8660$ (six terms) ($\frac{\sqrt{3}}{2} \approx 0.8660$)



In all cases we get (just from the first six terms) something very close to $\sin x$.

Section 3.2 : Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 56

A sequence is an infinite ordered list

$$a_1, a_2, a_3, \dots$$

- The items in list a_1, a_2 etc. are called terms (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \dots$ can be denoted by (a_n) or by $(a_n)_{n=1}^{\infty}$.

  the way (1,2) is used for the ordered pair (1,2) [as opposed to {1,2}]
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0$, $a_2 = (-1)^2 + 1 = 2$, $a_3 = (-1)^3 + 1 = 0, \dots$
   $$0, 2, 0, 2, 0, 2, \dots$$
2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1$, $a_2 = \sin(\pi) = 0$, $a_3 = \sin(\frac{3\pi}{2}) = -1$, $a_4 = \sin(2\pi) = 0, \dots$.
   $$1, 0, -1, 0, 1, 0, -1, 0, \dots$$
3. $(\frac{1}{n}\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1$, $a_2 = \frac{1}{2}\sin(\pi) = 0$, $a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3}$, $a_4 = \frac{1}{4}\sin(2\pi) = 0, \dots$.
   $$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \dots$$

Visualising a sequence

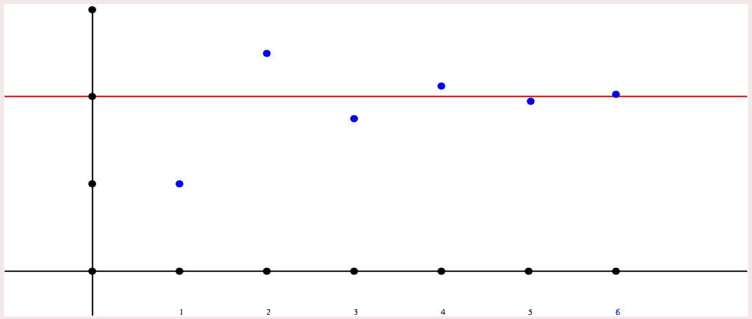
One way of visualizing a sequence is to consider is as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

Example 57

$(2 + (-1)^n 2^{1-n})_{n=1}^{\infty}$. *Write* $a_n = 2 + (-1)^n 2^{1-n}$. *Then*

$$a_1 = 2 - 2^0 = 1,\ a_2 = 2 + 2^{-1} = \frac{5}{2},\ a_3 = 2 - 2^{-2} = \frac{7}{4},\ a_4 = 2 + 2^{-3} = \frac{17}{8}.$$

Graphical representation of (a_n)*:*



The sequence $\left(2+(-1)^n\frac{1}{2^{n-1}}\right)_{n=1}^{\infty}$

As n gets very large the positive number $\frac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\frac{1}{2^{n-1}}$ as small as we like.

Hence for very large values of n, the number $2+(-1)^n\frac{1}{2^{n-1}}$ is very close to 2. By taking n as large as we like, we can make this number as close to 2 as we like.

We say that the sequence converges to 2, or that 2 is the limit of the sequence, and write

$$\lim_{n\to\infty}\left(2+(-1)^n\frac{1}{2^{n-1}}\right)=2.$$

Note: Because $(-1)^n$ is alternately positive and negative as n runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

Convergence of a sequence : “official” definitions

Definition 58

The sequence (a_n) converges to the number L (or has limit L) if for every positive real number ε (no matter how small) there exists a natural number N with the property that the term a_n of the sequence is within ε of L for all terms a_n beyond the Nth term. In more compact language :

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon \ \forall n > N.$$

Notes

- If a sequence has a limit we say that it converges or is convergent. If not we say that it diverges or is divergent.
- If a sequence converges to L, then no matter how small a radius around L we choose, there is a point in the sequence beyond which all terms are within that radius of L. So beyond this point, all terms of the sequence are *very close together* (and very close to L). Where that point is depends on how you interpret “very close together”.

Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 59

1. $(\max\{(-1)^n, 0\})_{n=1}^{\infty}$: $0,\ 1,\ 0,\ 1,\ 0,\ 1, \dots$
   This sequence alternates between 0 and 1 and does not approach any limit.
2. *A sequence can be divergent by having terms that increase (or decrease) without limit.*
   $(2^n)_{n=1}^{\infty}$: $2,\ 4,\ 8,\ 16,\ 32,\ 64, \dots$
3. *A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose nth term is the nth digit after the decimal point in the decimal representation of π.*

Convergence is a precise concept!

Remark: The notion of a convergent sequence is sometimes described informally with words like "the terms get closer and closer to L as n gets larger". It is not true however that the terms in a sequence that converges to a limit L must get progressively closer to L as n increases.

Example 60

The sequence (a_n) is defined by

$$a_n = 0 \text{ if } n \text{ is even, } a_n = \frac{1}{n} \text{ if } n \text{ is odd.}$$

This sequence begins :

$$1,\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{5},\ 0,\ \frac{1}{7},\ 0,\ \frac{1}{9},\ 0, \dots$$

It converges to 0 although it is not true that every step takes us closer to zero.