Section 2.4: $\mathbb{R}$ is uncountable

Our goal in this section is to show that the set $\mathbb{R}$ of real numbers is uncountable or non-denumerable; this means that its elements cannot be listed, or cannot be put in bijective correspondence with the natural numbers.

We saw at the end of Section 2.3 that $\mathbb{R}$ has the same cardinality as the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, or the interval $(-1, 1)$, or the interval $(0, 1)$. We will show that the open interval $(0, 1)$ is uncountable.

$(0,1)$ is uncountable

This assertion and its proof date back to the 1890's and to Georg Cantor. The proof is often referred to as "Cantor's diagonal argument" and applies in more general contexts than we will see in these notes.
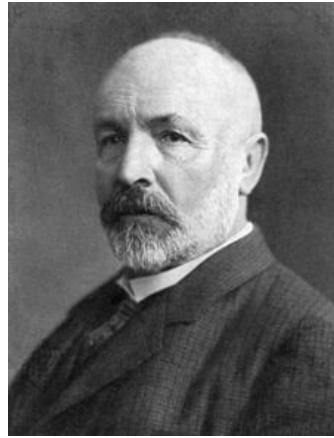
Georg Cantor : born in St Petersburg (1845), died in Halle (1918)

Theorem 44

The open interval $(0,1)$ *is not a countable set.*

The open interval $(0, 1)$ is not a countable set

We recall precisely what this set is.

- It consists of all real numbers that are greater than zero and less than 1, or equivalently of all the points on the number line that are to the right of 0 and to the left of 1.

- It consists of all numbers whose decimal representation have only 0 before the decimal point (except $0.000\ldots$ which is equal to 0, and $0.99999\ldots$ which is equal to 1).

- Note that the digits after the decimal point may terminate in an infinite string of zeros, or may have a repeating pattern to their digits, or may not have either of these properties. The interval $(0, 1)$ includes all these possibilities.

A hypothetical bijective correspondence

Our goal is to show that the interval $(0, 1)$ cannot be put in bijective correspondence with the set $\mathbb{N}$ of natural numbers. Our strategy is to show that no attempt at constructing a bijective correspondence between these two sets can ever be complete; it can never involve *all* the real numbers in the interval $(0, 1)$ no matter how it is devised.

So imagine that we had a listing of the elements of the interval $(0, 1)$. Such a correspondence would have to look something like the following.

$\mathbb{N}$		$(0, 1)$
1	$\longleftrightarrow$	$0.13567324\ldots$
2	$\longleftrightarrow$	$0.10000000\ldots$
3	$\longleftrightarrow$	$0.32323232\ldots$
4	$\longleftrightarrow$	$0.56834662\ldots$
5	$\longleftrightarrow$	$0.79993444\ldots$
$\vdots$		$\vdots$

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0, 1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
$\vdots$		$\vdots$

Look at the first digit after the decimal point in Item 1 in the list. If this is 1, write 2 as the first digit after the decimal point in x. Otherwise, write 1 as the first digit after the decimal point in x. So x differs in its first digit from Item 1 in the list.

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0,1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
$\vdots$		$\vdots$

Look at the second digit after the decimal point in Item 2 in the list. If this is 1, write 2 as the second digit after the decimal point in x. Otherwise, write 1 as the second digit after the decimal point in x. So x differs in its second digit from Item 2 in the list.

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0, 1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
⋮		⋮

Look at the third digit after the decimal point in Item 3 in the list. If this is 1, write 2 as the third digit after the decimal point in x. Otherwise, write 1 as the third digit after the decimal point in x. So x differs in its third digit from Item 3 in the list.

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain **every** sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0, 1)$
1	$\longleftrightarrow$	$0.\mathbf{1}3567324\ldots$
2	$\longleftrightarrow$	$0.1\mathbf{0}000000\ldots$
3	$\longleftrightarrow$	$0.32\mathbf{3}23232\ldots$
4	$\longleftrightarrow$	$0.568\mathbf{3}4662\ldots$
5	$\longleftrightarrow$	$0.7999\mathbf{3}444\ldots$
$\vdots$		$\vdots$

Continue to construct x digit by digit in this manner. At the nth stage, look at the nth digit after the decimal point in Item n in the list. If this is 1, write 2 as the nth digit after the decimal point in x. Otherwise, write 1 as the nth digit after the decimal point in x. So x differs in its nth digit from Item n in the list.

Cantor's Diagonal Argument

What this process constructs is an element x of the interval $(0,1)$ that does not appear in the proposed list. The number x is not Item 1 in the list, because it differs from Item 1 in its 1st digit, it is not Item 2 in the list because it differs from Item 2 in its 2nd digit, it is not Item n in the list because it differs from Item n in its nth digit.

Note:

- In our example, the number x would start $0.21111\ldots$.

- According to our construction, our x will always have all its digits equal to 1 or 2. So not only have we shown that the interval $(0,1)$ is uncountable, we have even shown that the set of all numbers in this interval whose digits are all either 1 or 2 is uncountable.

- A challenging exercise : why would the same proof not succeed in showing that the set of rational numbers in the interval $(0,1)$ is uncountable?

More on Cantor

Informally, Cantor's diagonal argument tells us that the "infinity" that is the cardinality of the real numbers is "bigger" than the "infinity" that is the cardinality of the natural numbers, or integers, or rational numbers. He was able to use the same argument to construct examples of infinite sets of different (and bigger and bigger) cardinalities. So he actually established the notion of infinities of different magnitudes.

The work of Cantor was not an immediate hit within his own lifetime. It met some opposition from the finitist school which held that only mathematical objects that can be constructed in a finite number of steps from the natural numbers could be considered to exist. Foremost among the proponents of this viewpoint was Leopold Kronecker.

Leopold Kronecker (1823-1891)

God made the integers, all else is the work of man.

What good your beautiful proof on π? Why investigate such problems, given that irrational numbers do not even exist?

Hilbert

Cantor had influential admirers too, among them David Hilbert, who set the course of much of 20th Century mathematics in his address to the International Congress of Mathematicians in Paris in 1900.

David Hilbert (1862-1943)

No one shall expel us from the paradise that Cantor has created for us.

What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

The Continuum Hypothesis

Hilbert's address to the Paris Congress is one of the most famous mathematical lectures ever. In it he posed 23 unsolved problems, the first of which was Cantor's Continuum Hypothesis.

The Continuum Hypothesis proposes that every subset of $\mathbb{R}$ is either countable (i.e. has the same cardinality as $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Q}$) or has the same cardinality as $\mathbb{R}$.

This seems like a question to which the answer should be either a straightforward yes or no.

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It took the work of Kurt Gödel in the 1930s and Paul Cohen in the 1960s to reach the conclusion that the answer to this question of Cantor is undecidable. This means essentially that the standard axioms of set theory do not provide enough structure to determine the answer to the question.

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Both the Continuum Hypothesis and its negation are consistent with the working rules of mathematics. People who work in set theory can legitimately assume that either the Continuum Hypothesis is satisfied or not. Fortunately most of us can get on with our mathematical work without having to worry about this very often.

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References for this stuff:

1. Reuben Hersh, *What is Mathematics, Really?* Oxford University Press, 1997
2. Ben Yandell, *The Honors Class*, A K Peters, 2002

Note on Exam Questions

From the Summer 2015 exam:

Q2 (a) Give an example of

(i) An infinite subset of $\mathbb{R}$ in which every element is negative.

(ii) A subset of $\mathbb{R}$ that is bounded above but not below.

(iii) A subset of $\mathbb{R}$ that is infinite, countable and bounded.

Learning Outcomes for Section 2.4

After studying this section you should be able to

- Use Cantor's diagonal argument to prove that the interval $(0, 1)$ is uncountable.

- Make a few remarks about the history of this discovery.