# Chapter 3

# Linear Transformations, Eigenvectors and Similarity

## 3.1 Changes of Basis

The dimension of the space $\mathbb{F}^n$ is $n$ - the *standard basis* consists of the column vectors $e_1, \dots, e_n$, where $e_i$ has 1 in position $i$ and zeros in all other positions.

For example, in $\mathbb{R}^3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the standard basis $\mathcal{E} = \{e_1, e_2, e_3\}$.

### 3.1.1 How can we recognize a basis of $\mathbb{F}^n$?

It should have $n$ elements, which should be column vectors in $\mathbb{F}^n$. But some sets of three column vectors in $\mathbb{R}^3$ are bases of $\mathbb{R}^3$ and some are not. How do we know?

**Theorem 3.1.1.** *Let* $B = \{v_1, \dots, v_n\}$ *be any set of* $n$ *vectors in* $\mathbb{F}^n$*. Then* $B$ *is a basis of* $\mathbb{F}^n$ *if and only if the matrix* $A$ *whose columns are* $v_1, \dots, v_n$ *has an inverse in* $M_n(\mathbb{F})$*.*

*Proof.* Suppose that $A$ has an inverse of in $M_n(\mathbb{F})$. Then $AA^{-1} = I_n$, and $Aw_1 = e_1$, where $w_1$ is the first column of $A^{-1}$. It follows that $e_1$ is a linear combination of $v_1, \dots, v_n$. Similarly each $e_i$ is in the linear span of $\{v_1, \dots, v_n\}$, and so $\{v_1, \dots, v_n\}$ is a spanning set of $\mathbb{F}^n$. Hence it is a basis of $\mathbb{F}^n$ by Lemma 2.3.1.

On the other hand, suppose that $B$ is a basis of $\mathbb{F}_n$. Then $e_1$ is a linear combination of the columns of $B$, and so $e_1 = Bw_1$, for some $w_1 \in \mathbb{F}^n$. Similarly $e_i = Bw_i$, for $i = 2, \dots, n$. It follows that $AW = I_n$, where $W$ is the matrix in $M_n(\mathbb{F})$ whose columns are $w_1, \dots, w_n$, and hence $A$ has an inverse in $M_n(\mathbb{F})$. $\square$

### 3.1.2 Moving between two bases - an example

Suppose we have another basis $\mathcal{B} = \{b_1, b_2, b_3\}$ of $R^3$ (besides the standard basis), where

$$b_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

You can check that $\mathcal{B}$ is linearly independent, hence is is a basis of $\mathbb{R}^3$ - for example by checking that the RREF of the $3 \times 3$ matrix $[b_1\ b_2\ b_3]$ is $I_3$. We write $B$ for the matrix with columns $b_1, b_2, b_3$.

**Question:** Suppose we have some other vector in $\mathbb{R}^3$, for example $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

*What are the coordinates of* $v$ *with respect to* $\mathcal{B}$*?*

**Another Question:** Why would we want to know this?
*Partial answer* - the standard basis is very useful for example for describing a rotation of $\mathbb{R}^3$ through 180° about the Z-axis. We can say exactly how this affects each of the standard basis vectors. But if we wanted to describe a rotation around a different axis, say for example one in the direction of $b_1$ (which is perpendicular to $b_2$ and $b_3$) maybe the standard basis is not the best for that. We will come back to this theme shortly, for now the suggestion is to just keep it in mind.

Back to the first question: if we knew how to write $e_1, e_2$ and $e_3$ as a linear combination of $b_1, b_2, b_3$, we could do the same for $v$ (or any vector). To figure this out: the $\mathcal{B}$-coordinates of $e_1$ are the values of $x, y, z$ in the unique solution of

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ or } B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = e_1.$$

The corresponding values are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which means they are the entries of Column 1 of $B^{-1}$. In the same way, the $\mathcal{B}$-coordinates of $e_1$ and $e_3$ are given by Columns 2 and 3 of $B^{-1}$.

We can confirm this for our example:

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \; B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

Looking at (for example) Column 2 of $B^{-1}$ we can confirm that its entries are the $\mathcal{B}$-coordinates of $e_2$:

$$1b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3 = 1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2} - \frac{1}{2} \\ 1 - \frac{1}{2} + \frac{1}{2} \\ -1 + 1 \end{bmatrix} = e_2.$$

Now for the $\mathcal{B}$-coordinates of $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. We write $[v]_{\mathcal{B}}$ for the column whose entries are the $\mathcal{B}$-coordinates of $v$. The punchline is that we can now achieve this through a matrix-vector product.

$$\begin{aligned} v = 2e_1 + 1e_2 + 3e_3 \Longrightarrow [v]_{\mathcal{B}} &= 2[e_1]_{\mathcal{B}} + 1[e_2]_{\mathcal{B}} + 3[e_3]_{\mathcal{B}} \\ &= \begin{bmatrix} [e_1]_{\mathcal{B}} & [e_2]_{\mathcal{B}} & [e_3]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

**Conclusion:** $v = 6b_1 + \frac{9}{2}b_2 + \frac{1}{2}b_3$.

**Exercise:** Confirm this conclusion by direct calculation.

**More important conclusion:** To find the $\mathcal{B}$-coordinates of *any* vector $v$ in $\mathbb{R}^3$, what we need to do is multiply $v$ on the left by the *change of basis* matrix from the standard basis to $\mathcal{B}$. This is the inverse of the matrix whose columns are the elements of $\mathcal{B}$ (written in the standard basis).

LEARNING OUTCOMES FOR THIS SECTION

1. *How to recognize when a set of $n$ column vectors in $\mathbb{R}^n$ (or $\mathbb{F}^n$) forms a basis.*
   Think about the statement of Theorem 3.1.1 first, and try out a few of your own examples with $n = 2$ or $n = 3$, to get a sense of what it is saying. This is a good thing to do before studying the proof (which is not necessarily urgent).

2. *To recognize that elements of $\mathbb{R}^n$ (or $\mathbb{F}^n$) have different coordinates with respect to different bases*
   Think about this for some examples with $n = 2$.

3. *To use the change of basis matrix to write the coordinates of any vector in $\mathbb{F}^n$ with respect to a given basis*
   The instruction for how to do this is the "More important conclusion" above. Try it out for other vectors besides the $v$ that was used here. Use a couple of such examples to satisfy yourself that it works, then go over the steps to think about *why* it works.