where we write E_{ij} for the matrix with 1 in the (i,j)-position and zero in all other positions. From the above, we see that $\{E_{12}-E_{21}, E_{13}-E_{31}, E_{23}-E_{32}\}$ is a spanning set of V. This set is also linearly independent: if

$$a(E_{12}-E_{21}) + b(E_{13}-E_{31}) + c(E_{23}-E_{32}) = 0_{3\times 3}$$

for some real scalars a, b, c, then by looking at the entries in the $(1,2), (1,3)$ and $(2,3)$ positions, we observe that $a = b = c = 0$. We conclude that $\{E_{12}-E_{21}, E_{13}-E_{31}, E_{23}-E_{32}\}$ is a basis of V and that $\dim V = 3$.

Exercise 2.2.10. *What is the dimension of the space of skew-symmetric matrices in $M_n(\mathbb{R})$? What about the space of symmetric matrices?*

2.3 More on Bases and Dimension

In this section we note a few more properties of bases of finite dimensional vector spaces. We start with a connection to matrices. We let V be a vector space of dimension n over a field $\mathbb{F}$. Recall that this means that every basis of V has n elements, and a basis of V is a linearly independent spanning set of B. This means that it is both a minimal spanning set of V and a maximal linearly independent subset of V.

Lemma 2.3.1. *Every linearly independent subset of V with n elements is a basis of V.*

Proof. Let $L = \{v_1, \dots, v_n\}$ be a linearly independent subset of V. If L is not a spanning set of V, then there is some $v \in V$ with $v \notin \langle L \rangle$. It follows that the set $L' = \{v_1, \dots, v_n, v\}$ is linearly independent in V, contrary to Theorem 2.2.6. □

Lemma 2.3.2. *Every spanning set of V with n elements is a basis of V.*

Proof. Let V be a spanning set of V with n elements. If V is not linearly independent, then V contains a proper subset that spans V but has fewer than n elements, contrary to Theorem 2.2.6. □

Lemma 2.3.3. *If L is a linearly independent subset of V, then L can be extended to a basis of V.*

Proof. Write $L = \{v_1, \dots, v_k\}$. Then $k \leqslant n$ by Theorem 2.2.6. If $k = n$, then L spans V by Lemma 2.3.1, and L is a basis of V. If $k < n$, then L is not a spaning set of V, and there is an element $v_{k+1} \in V$ with $v_{k+1} \notin \langle L \rangle$. Then $\{v_1, \dots, v_k, v_{k+1}\}$ is linearly independent. We may continue in this way to add elements from outside the existing span, until we reach a basis of V after $n-k$ steps. □

For any field $\mathbb{F}$, $\mathbb{F}^n$ denotes the space of all column vectors with n entries. The *standard basis* of $\mathbb{F}^n$ is $\{e_1, \dots, e_n\}$, where e_i has 1 in position i and 0 in all other positions. That $\{e_1, \dots, e_n\}$ is linearly independent and spans $\mathbb{F}^n$ can be confirmed from the relevant definitions.

Theorem 2.3.4. *Let $B = \{v_1, \dots, v_n\}$ be any set of n vectors in $\mathbb{F}^n$. Then B is a basis of $\mathbb{F}^n$ if and only if the matrix A whose columns are $v_1, \dots, v_n$ has an inverse in $M_n(\mathbb{F})$.*

Proof. Suppose that A has an inverse of in $M_n(\mathbb{F})$. Then $AA^{-1} = I_n$, and $Aw_1 = e_1$, where w_1 is the first column of A^{-1}. It follows that e_1 is a linear combination of $v_1, \dots, v_n$. Similarly each e_i is in the linear span of $\{v_1, \dots, v_n\}$, and so $\{v_1, \dots, v_n\}$ is a spanning set of $\mathbb{F}^n$. Hence it is a basis of $\mathbb{F}^n$ by Lemma 2.3.1.

On the other hand, suppose that B is a basis of $\mathbb{F}_n$. Then e_1 is a linear combination of the columns of B, and so $e_1 = Bw_1$, for some $w_1 \in \mathbb{F}^n$. Similarly $e_i = Bw_i$, for $i = 2, \dots, n$. It follows that $AW = I_n$, where W is the matrix in $M_n(\mathbb{F})$ whose columns are $w_1, \dots, w_n$, and hence A has an inverse in $M_n(\mathbb{F})$. □

We finish (for now) on this topic by noting that $\mathbb{F}^n$ is the generic and even (sort of) the only vector space of dimension n over $\mathbb{F}$. Suppose that V is a $\mathbb{F}$-vector space with $\dim V = n$, and let $B = \{v_1, \dots, v_n\}$ be a basis of V over $\mathbb{F}$. For every element $v \in V$, there is a unique expression for v as a linear combination of the elements of B:

$$v = a_1 v_1 + \dots + a_n v_n.$$

We refer to $a_1, \dots, a_n$ as the *coordinates* of v with respect to the basis B. With this association, we can consider v to be represented by the column vector in $\mathbb{F}^n$ whose entries are $a_1, \dots, a_n$. This association defines a bijective correspondence between V and $\mathbb{F}^n$, and means that we can identify these two vector spaces as being essentially the same. Different bases of V correspond to different identifications of V with $\mathbb{F}^n$, and we will explore the realtionships between these in Chapter 3.