We could forget about the third element of S and just write

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a+b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a)\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So the three elements of S are not necessary to form a spanning set of $\mathbb{R}^2$. We could span $\mathbb{R}^2$ just with the subset $\{\binom{2}{1}, \binom{3}{1}\}$ of S. We note that $\binom{1}{-1}$ is a $\mathbb{R}$-linear combination of the other two elements of S. If we drop this element from S, we can still recover it in the span of the remaining elements.

The second example above motivates this lemma, which we will explore further in the next section.

Lemma 2.1.4. *Suppose that $S_1 \subset S$, where S is a subset of a vector space V. Then $\langle S_1 \rangle \subseteq \langle S \rangle$, and $\langle S_1 \rangle = \langle S \rangle$ if and only if every element of $S \backslash S_1$ is a linear combination of elements of S_1.*

We finish this section by noting the distinction between a *finite dimensional* and *infinite dimensional* vector space.

Definition 2.1.5. *A vector space is said to be* finite dimensional *if it has a finite spanning set. A vector space that does not have a finite spanning set is* infinite dimensional*.*

Two examples of infinite dimensional vector spaces

1. The vector space $\mathbb{R}[x]$ of all polynomials with real coefficients is infinite dimensional. To see this, let S be any finite subset of $\mathbb{R}[x]$ (i.e. a finite set of polynomials). Let x^k be the highest power of x to appear in any element of S. Then no linear combination of elements of S has degree exceeding x^k, so the linear span of S cannot be all of $\mathbb{R}[x]$.

2. The set $\mathbb{R}$ of real numbers is infinite dimensional as a vector space over the field $\mathbb{Q}$ of rational numbers.

2.2 Linear Independence

Definition 2.2.1. *Let S be a subset of a vector space V, having at least 2 elements. Then S is* linearly independent *if no element of S is a linear combination of the other elements of S (equivalently, if no element of S belongs to the span of the other elements of S).*

A subset consisting of a single element is linear independent, provided that its unique element is not the zero vector.

To decide if a given set is linearly independent, Definition 2.2.1 is maybe not the most useful formulation, because it requires us to check something separately for each element of S, which could take a lot of work. The following altenative version of the definition might have less appeal at an intuitive level, but it is often more useful in practice.

Definition 2.2.2. *Let S be a non-empty subset of a vector space V. Then S is* linearly independent *if the only way to write the zero vector in V as a linear combination of elements of S is to take all the coefficients to be 0.*

Equivalence of the two definitions
Let $S = \{v_1, \dots, v_k\}$ and suppose that $v_1 \in \langle v_2, \dots, v_k \rangle$. Then

$$v_1 = a_2 v_2 + \cdots + a_k v_k,$$

and

$$0 = -v_1 + a_2 v_2 + \cdots + a_k v_k$$

is an expression for the zero vector as a linear combination of elements of S, whose coefficients are not all zero.

On the other hand, suppose that

$$0 = c_1 v_1 + \cdots + c_k v_k,$$

where the scalars c_i are not all zero. If $c_1 \neq 0$ (for example), then the above equation can be rearranged to express v_1 as a linear combination of $v_2, \ldots, v_k$:

$$v_1 = -\frac{c_2}{c_1} v_2 - \cdots - \frac{c_k}{c_1} v_k.$$

Example 2.2.3. *In* $\mathbb{R}^3$, *let* $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} \right\}$.

To determine whether S is linearly independent, we must investigate whether the system of equations

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solutions other than $(x, y, z) = (0, 0, 0)$. The augmented matrix of this system, and its RREF, are

$$\begin{bmatrix} 1 & -2 & -3 & 0 \\ 2 & 3 & 8 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus for any t, $(x, y, z) = (-t, -2t, t)$ is a solution, and for example by taking $t = 1$ we see that

$$-1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and hence that each of the three elements of S is a linear combination of the other two. So S is not linearly independent (we say that S is *linearly dependent*).

We note below some characterizations of linearly independent set. Let S be a subset of a vector space V.

1. S is linearly independent if S is a *minimal* spanning set of its linear span - no proper subset of S spans the same subspace of V that S does, or every proper subspace of S spans a strictly smaller subspace than S itself.

2. S is linearly independent if every element of $\langle S \rangle$ has a *unique* expression as a linear combination of elements of S. If a particular element of $\langle S \rangle$ had two different expressions as linear combinations of S, with different coefficients, then subtracting one from the other would give a non-trivial expression for 0_V as a linear combination of elements of S, and by Definition 2.2.2 we would conclude that S is linearly dependent.

3. Another version of 2. above: S is linearly independent if every element of the span of S has *unique coordinates* in terms of the elements of S.

So a linearly independent set in a vector space V is a *minimal* or *irredundant* spanning set for its linear span. If its linear span happens to be all of V, it gets a special name.

Definition 2.2.4. *A* basis *of a vector space* V *is a spanning set of* V *that is linearly independent. [Plural: bases]*

Lemma 2.2.5. *If* S *is a finite spanning set of a vector space* V, *then* S *contains a basis of* V.

Proof. If S is not linearly independent, then some element v_1 of S is in the span of the other elements of S, and $S_1 := S \backslash \{v_1\}$ is again a spanning set of V. If S_1 is not linearly independent, then we can discard an element of S_1 that is in the linear span of the others, to form a smaller spanning set S_2. Since S is finite, this process cannot continue indefinitely, and it concludes with a linearly independent spanning set of V. □

We will show that if V has a finite basis, then *every* basis has the same number of elements. This number is then referred to as the *dimension* of V. The key to this is to show that the number of elements in *any* spanning set of V is an upper bound for the number of elements in *any* linearly independent subset of V. The following theorem

Theorem 2.2.6. *Let V be a vector space over a field $\mathbb{F}$, and suppose that $S = \{v_1, \dots, v_n\}$ is a spanning set of V. Then the number of elements in a linearly independent subset of V cannot exceed n.*

Proof. Let $S = \{v_1, \dots, v_t\}$ be a *spanning set* of V. Let $L = \{y_1, \dots, y_k\}$ be a linearly independent subset of V. We need to show $k \leqslant t$.

1. We know that $y_1 \in V$, so y_1 is a linear combination of elements of S. After reordering the elements of S if necessary, we may assume that v_1 appears with a non-zero coefficient in an expression for y_1 as a linear combination of $v_1, \dots, v_n$. It follows that v_1 belongs to the linear span of $\{y_1, v_2 \dots, v_t\}$, and hence that
$$S_1 = \{y_1, v_2, \dots, v_t\}$$
is again a spanning set of V.

2. Then y_2 is a linear combination of the elements of S_1, and it is not just a scalar multiple of y_1, since L is linearly independent. So one of $v_2, \dots, v_t$ is involved (with non-zero coefficient) in an expression for y_2 as a linear combination of $y_1, v_2, \dots, v_t$. After relabelling if necessary, we can suppose that v_2 has this property. Then v_2 is a linear combination of $y_1, y_2, v_3, \dots, v_n$, and
$$S_2 = \{y_1, y_2, v_3, \dots, v_t\}$$
is a spanning set of V.

3. Continuing in this way (and relabelling the elements of S) at each step if necessary, we adjust the spanning set by replacing v_3 with y_3, then v_4 wth y_4, and so on. Since $\{v_1, \dots, v_{k-1}\}$ is not a spanning set of V, there is still at least one element of the orignal S left after $k-1$ steps, at which point we have $S_{k-1} = \{v_1, v_2, \dots, v_{k-1}, y_k, \dots, y_t\}$.

We conclude that $t \geqslant k$, and the number of elements in a linearly independent subset of V cannot exceed the number in a spanning set. ☐

Theorem 2.2.6 is often referred to as the *Steinitz exchange lemma*, and it is the most important technical ingredient in establishing the following crucial property of finite dimensional vector spaces V.

Theorem 2.2.7. *If V is a finite dimensional vector space over a field $\mathbb{F}$, then every basis of V has the same number of elements.*

Proof. Let B_1 and B_2 be bases of V. Then B_1 is linearly independent and B_2 is a spanning set of V, so $|B_1| \leqslant |B_2|$ by Theorem 2.2.6. Also, B_2 is linearly independent and B_1 is a spanning set of V, so $|B_2| \leqslant |B_1|$ by Theorem 2.2.6. Hence $|B_1| = |B_2|$. ☐

Definition 2.2.8. *The number of elements in any (hence every) basis of a finite dimensional vector space V is called the* dimension *of V, denoted* $\dim V$.

Example 2.2.9. Let V be the space of skew-symmetric matrices in $M_3(\mathbb{R})$ (a matrix A is *skew-symmetric* if $A^T = -A$). Then
$$V = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The typical element of V noted above can be written as
$$\begin{aligned} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} &= a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}), \end{aligned}$$