Chapter 2

Vector Spaces and Linear Transformations

2.1 Subsapces and spanning sets

We think of the real number line $\mathbb{R}$ as begin "1-dimensional", and of $\mathbb{R}^2$ as being "2-dimensional" and of $\mathbb{R}^3$ as being 3-dimensional. These terms are used not only in mathematics but in everyday language aswell. In linear algebra, they mean something quite precise, that also applies much more generally. To say that $\mathbb{R}$ is 1-dimensional means that we only need one real number to specify the position of a point in $\mathbb{R}$. For a point in $\mathbb{R}^2$, we need to specify two real numbers, for example its x and y coordinates - but these are not the only options. We could use its distance from the origin, and the angle that the line segment joining it to the orgin makes with the positive X-axis. We could specify its position relative to another pair of lines, instead of the two coordinate axes. A 2-dimensional space can be viewed as 1-dimensional space augmented with another independent copy of itself. The x-coordinate of a point in $\mathbb{R}^2$ can be any real number. But once that has been specified - say it is 2 - then the range of possibilities for the point includes *all* points in $\mathbb{R}^2$ that lie on the line $x = 2$, all points with x-coordinate 2. The set of such points forms another copy of $\mathbb{R}$ inside $\mathbb{R}^2$, it is the vertical line through $(2, 0)$.

Another example of a vector space that is 2-dimensional is the space V consisting of all symmetric 2×2 matrices in $M_2(\mathbb{R})$ with trace zero. A symmetric matrix is one that is equal to its transpose. Trace zero means the sum of the entries on the main diagoanl is zero. So a symmetric matrix of trace zero in $M_2(\mathbb{R})$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, where a and b may be any real numbers and may be chosen independently. So it takes a choice of two real numbers to specify an element of V. Also,

$$V = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\} = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

We say that the set $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is a *spanning set* of V over $\mathbb{R}$.

Definition 2.1.1. *Let V be a vector space over a field $\mathbb{F}$. A subset U of V is a* subspace *(or* vector subspace*) of V if U is itself a vector space over $\mathbb{F}$, under the addition and scalar multiplication operations of V.*

Two things need to be checked to confirm that a subset U of a vector space V is a *subspace*:

1. That U is *closed* under the addition in V: that $u_1 + u_2 \in U$ whenever $u_1 \in U$ and $u_2 \in U$;
2. That U is *closed* under scalar multiplication: that $\alpha u \in U$ whenever $u \in U$ and $\alpha \in \mathbb{F}$.

Examples

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are rational polynomials of degree at most 2, then so also is $f(x) + g(x)$. If $f(x)$ is a rational polynomial of degree at least 2, then so is $\alpha f(x)$ for any $\alpha \in \mathbb{Q}$.

2. The set of $\mathbb{C}$ complex numbers is a vector space over the set of real numbers. Within $\mathbb{C}$, the subset $\mathbb{R}$ is an example of a vector subspace over $\mathbb{R}$. An example of a subset of $\mathbb{C}$ that is *not* a real vector subset is the unit circle S in the complex plane - this is the set of complex numbers of modulus 1, it consists of all complex numbers of the form $a + bi$, where $a^2 + b^2 = 1$. This is closed neither under additon nor multiplication by real scalars.

3. The Cartesian plane $\mathbb{R}^2$ is a real vector space. Within $\mathbb{R}^2$, let $U = \{(a, b) : a \geqslant 0, b \geqslant 0\}$. Then U is closed under addition and under multiplication by positive scalars. It is not a vector subspace of $\mathbb{R}^2$, because it is not closed under multiplication by negative scalars.

4. Let v be a (fixed) non-zero vector in $\mathbb{R}^3$, and let $v^\perp = \{u \in \mathbb{R}^3 : u^T v = 0\}$. Then $v^\perp$ is not empty since $0 \in v^\perp$. Suppose that $u_1, u_2 \in v^\perp$. Then $(u_1 + u_2)^T v = (u_1^T + u_2^T)v = u_1^T v + u_2^T v = 0$. So $u_1 + u_2 \in v^\perp$ and $v^\perp$ is closed under addition. If $u \in v^\perp$ and $\alpha \in \mathbb{R}$, then $(\alpha u)^T v = \alpha u^T v = \alpha 0 = 0$, and $\alpha u \in v^\perp$. Hence $v^\perp$ is closed under scalar multiplication in $\mathbb{R}^3$. Conclusion: $v^\perp$ is a vector subspace of $\mathbb{R}^3$. Note that $v^\perp$ is not all of $\mathbb{R}^3$, since $v \notin v^\perp$.

Definition 2.1.2. *Let V be a vector space over a field $\mathbb{F}$, and let S be a non-empty subset of V. The* $\mathbb{F}$-linear span *(or just* span*) of S, denoted $\langle S\rangle$ is the set of all $\mathbb{F}$-linear combinations of elements of S in V. If $S = V$, then S is called a* spanning set *of V. This means that every element of V is a linear combination of elements of S.*

For a subset S of a F-vector space V, the sum of any two linear combinations of elements of S is an element of S, and any scalar multiple of a linear combination of elements of S is again a linear combination of elements of S; hence the following lemma.

Lemma 2.1.3. *For any subset S of a vector space V, $\langle S\rangle$ is a subspace of V.*

Examples

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$ (this means that P_2 is itself a vector space under the addition and scalar mutliplication operations of $\mathbb{Q}[x]$). If $S = \{x^2 + 1, x + 1\}$, then
$$\langle S\rangle = \{a(x^2 + 1) + b(x + 1) : a, b \in \mathbb{Q}\} = \{ax^2 + bx + a + b : a, b \in \mathbb{Q}\}.$$
So $\langle S\rangle$ consists of all rational polynomials of degree at most 2, in which the constant coefficient is the sum of the coefficients of x and x^2. For example, $x^2 + 2x + 3 \in \langle S\rangle$ but $x^2 + 2x + 4 \notin \langle S\rangle$. Since $\langle S\rangle$ does not include all elements of P_2, S is not a spanning set of P_2 over $\mathbb{Q}$.

2. The set $S = \left\{\binom{3}{1}, \binom{2}{1}, \binom{1}{-1}\right\}$ is a spanning set of the vector space $\mathbb{R}^2$ of all real column vectors with two entries. If $v = \binom{a}{b} \in \mathbb{R}^2$, we can write v as a linear combination of the elements of S, for example by writing
$$\begin{pmatrix} a \\ b \end{pmatrix} = (a + b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a - b)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - b\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
This is not the only way to do it. We could also write
$$\begin{pmatrix} a \\ b \end{pmatrix} = (4a + b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-5a - b)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-a - b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We could forget about the third element of S and just write

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a+b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a)\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So the three elements of S are not necessary to form a spanning set of $\mathbb{R}^2$. We could span $\mathbb{R}^2$ just with the subset $\{\binom{2}{1}, \binom{3}{1}\}$ of S. We note that $\binom{1}{-1}$ is a $\mathbb{R}$-linear combination of the other two elements of S. If we drop this element from S, we can still recover it in the span of the remaining elements.

The second example above motivates this lemma, which we will explore further in the next section.

Lemma 2.1.4. *Suppose that* $S_1 \subset S$*, where* S *is a subset of a vector space* V*. Then* $\langle S_1 \rangle \subseteq \langle S \rangle$*, and* $\langle S_1 \rangle = \langle S \rangle$ *if and only if every element of* $S \backslash S_1$ *is a linear combination of elements of* S_1.

We finish this section by noting the distinction between a *finite dimensional* and *infinite dimensional* vector space.

Definition 2.1.5. *A vector space is said to be* finite dimensional *if it has a finite spanning set. A vector space that does not have a finite spanning set is* infinite dimensional*.*

Two examples of infinite dimensional vector spaces

1. The vector space $\mathbb{R}[x]$ of all polynomials with real coefficients is infinite dimensional. To see this, let S be any finite subset of $\mathbb{R}[x]$ (i.e. a finite set of polynomials). Let x^k be the highest power of x to appear in any element of S. Then no linear combination of elements of S has degree exceeding x^k, so the linear span of S cannot be all of $\mathbb{R}[x]$.

2. The set $\mathbb{R}$ of real numbers is infinite dimensional as a vector space over the field $\mathbb{Q}$ of rational numbers.