Lecture 9: Subspaces

Definition Let V be a vector space over a field $\mathbb{F}$. A subset U of V is a subspace (or vector subspace) of V if U is itself a vector space over $\mathbb{F}$, under the addition and scalar multiplication operations of V.
Two things need to be checked to confirm that a subset U of a vector space V is a *subspace*:

1. That U is *closed* under the addition in V: that $u_1 + u_2 \in U$ whenever $u_1 \in U$ and $u_2 \in U$;
2. That U is *closed* under scalar multiplication: that $\alpha u \in U$ whenever $u \in U$ and $\alpha \in \mathbb{F}$.

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are rational polynomials of degree at most 2, then so also is $f(x) + g(x)$. If $f(x)$ is a rational polynomial of degree at least 2, then so is $\alpha f(x)$ for any $\alpha \in \mathbb{Q}$.
2. The set of $\mathbb{C}$ complex numbers is a vector space over the set of real numbers. Within $\mathbb{C}$, the subset $\mathbb{R}$ is an example of a vector subspace over $\mathbb{R}$. An example of a subset of $\mathbb{C}$ that is *not* a real vector subset is the unit circle S in the complex plane - this is the set of complex numbers of modulus 1, it consists of all complex numbers of the form $a + bi$, where $a^2 + b^2 = 1$. This is closed neither under additon nor multiplication by real scalars.

3. The Cartesian plane $\mathbb{R}^2$ is a real vector space. Within $\mathbb{R}^2$, let $U = \{(a, b) : a \geq 0, b \geq 0\}$. Then U is closed under addition and under multiplication by positive scalars. It is not a vector subspace of $\mathbb{R}^2$, because it is not closed under multiplication by negative scalars.
4. Let v be a (fixed) non-zero vector in $\mathbb{R}^3$, and let

$$v^{\perp} = \{u \in \mathbb{R}^3 : u^T v = 0\}.$$

Then $v^{\perp}$ is not empty since $0 \in v^{\perp}$. Suppose that $u_1, u_2 \in v^{\perp}$. Then

$$(u_1 + u_2)^T v = (u_1^T + u_2^T)v = u_1^T v + u_2^T v = 0.$$

So $u_1 + u_2 \in v^{\perp}$ and $v^{\perp}$ is closed under addition.
If $u \in v^{\perp}$ and $\alpha \in \mathbb{R}$, then $(\alpha u)^T v = \alpha u^T v = \alpha 0 = 0$, and $\alpha u \in v^{\perp}$. Hence $v^{\perp}$ is closed under scalar multiplication in $\mathbb{R}^3$.
Conclusion: $v^{\perp}$ is a vector subspace of $\mathbb{R}^3$. Note that $v^{\perp}$ is not all of $\mathbb{R}^3$, since $v \notin v^{\perp}$.

The linear span of a set

Definition Let V be a vector space over a field $\mathbb{F}$, and let S be a non-empty subset of V. The $\mathbb{F}$-*linear span* (or just *span*) of S, denoted $\langle span \rangle$ is the set of all $\mathbb{F}$-linear combinations of elements of S in V. If $S = V$, then S is called a *spanning set* of V. This means that every element of V is a linear combination of elements of S.

Lemma If S is a subset of a vector space V, then $\langle S \rangle$ is a subspace of V, and it is the smallest subspace of V that contains the set S.

Example

Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subspace consisting of all polynomials of degree at most 2,

$$P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}.$$

If $S = \{x^2 + 1, x + 1\}$, then

$$\langle S \rangle = \{a(x^2 + 1) + b(x + 1) : a, b \in \mathbb{Q}\} = \{ax^2 + bx + a + b : a, b \in \mathbb{Q}\}.$$

So $\langle S \rangle$ consists of all rational polynomials of degree at most 2, in which the constant coefficient is the sum of the coefficients of x and x^2. For example, $x^2 + 2x + 3 \in \langle S \rangle$ but $x^2 + 2x + 4 \notin \langle S \rangle$.