Section 1.4: Connections to Matrix Algebra

Elementary row operations may themselves be interpreted as matrix multiplication exercises.

- We write I_m for the $m \times m$ identity matrix
- We write $E_{i,j}$ for the matrix that has 1 in the (i,j)-position and zeros everywhere else.

Theorem 11

Let A be a $m \times m$ matrix. Then elementary row operations on A amount to multiplying A on the left by $m \times m$ matrices, as follows:

1. *Mutiplying Row i by the non-zero scalar α is equivalent to multiplying A on the left by the matrix $I_m + (\alpha - 1)E_{i,i}$.*
2. *Switching Rows i and k amounts to multiplying A on the left by the matrix $I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,i}$.*
3. *Adding $\alpha\times$ Row i to Row k amounts to multiplying A on the left by the matrix $I_m + \alpha E_{k,i}$.*

Elementary Row Operations as Matrix Multiplication

Matrices of the three types described in Theorem 11 are sometimes referred to as *elementary matrices*. They are always invertible, and their inverses are also elementary matrices. The statement that every matrix can be reduced to RREF through a sequence of EROs is equivalent to saying that for every matrix A with m rows, there exists a $m \times m$ matrix B, which is a product of elementary matrices, with the property that BA is in RREF.

Exercise 12

Write down the inverse of an elementary matrix of each of the three types, and show that it is also an elementary matrix.
(Hint: Think about how to reverse an elementary row operation, with another elementary row operation).

Exercise 13

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Using Gauss-Jordan elimination to calculate matrix inverses

Suppose that $A \in M_n(\mathbb{F})$, for some field $\mathbb{F}$. If A is invertible, let $v_1, v_2, \dots, v_n$ be the columns of its inverse. Then

$$AA^{-1} = A \begin{bmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{bmatrix} = A \begin{bmatrix} | & | & \dots & | \\ Av_1 & Av_2 & \dots & Av_n \\ | & | & \dots & | \end{bmatrix} = I_n.$$

For each i, Av_i is the ith column of the identity matrix, which has 1 in position i and zeros elsewhere. This means that v_i is the solution of the linear system $Av_i = e_i$, where e_i is column i of the identity matrix, and the variables are the unknown entries of v_i.
We need to this for each column, but we can combine this into a single process by writing $e_1, e_2, \dots, e_n$ as n distinct columns in the "right hand side" of a $n \times 2n$ augmented matrix.

Example of inverse calculation

Example Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

To calculate A^{-1}, We apply Gauss-Jordan elimination to the 3×6 matrix below

$$A' = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

We conclude

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}.$$

We think of the real number line $\mathbb{R}$ as begin "1-dimensional", and of $\mathbb{R}^2$ as being "2-dimensional" and of $\mathbb{R}^3$ as being 3-dimensional. These terms are used not only in mathematics but in everyday language aswell. In linear algebra, they mean something quite precise.

To say that $\mathbb{R}$ is 1-dimensional means that we only need one real number to specify the position of a point in $\mathbb{R}$.

For a point in $\mathbb{R}^2$, we need to specify two real numbers, for example its x and y coordinates - but these are not the only options. We could use its distance from the origin, and the angle that the line segment joining it to the orgin makes with the positive X-axis. We could specify its position relative to another pair of lines, instead of the two coordinate axes.

Another example of a vector space that is 2-dimensional is the space V consisting of all symmetric 2×2 matrices in $M_2(\mathbb{R})$ with trace zero. A symmetric matrix is one that is equal to its transpose. Trace zero means the sum of the entries on the main diagonal is zero.

Subspaces

Definition Let V be a vector space over a field $\mathbb{F}$. A subset U of V is a subspace (or vector subspace) of V if U is itself a vector space over $\mathbb{F}$, under the addition and scalar multiplication operations of V.

Two things need to be checked to confirm that a subset U of a vector space V is a *subspace*:

1. That U is *closed* under the addition in V: that $u_1 + u_2 \in U$ whenever $u_1 \in U$ and $u_2 \in U$;
2. That U is *closed* under scalar multiplication: that $\alpha u \in U$ whenever $u \in U$ and $\alpha \in \mathbb{F}$.

Examples

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are rational polynomials of degree at most 2, then so also is $f(x) + g(x)$. If $f(x)$ is a rational polynomial of degree at least 2, then so is $\alpha f(x)$ for any $\alpha \in \mathbb{Q}$.