Lecture 7: Possible outcomes of solving linear systems

1. The system may be inconsistent. This happens if a REF obtained from the augmented matrix has a leading 1 in its rightmost column.
2. The system may be consistent. Then one of the following occurs :
    1. There may be a unique solution. This happens if all variables are leading variables. In the case the RREF has the following form :

    $$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

       with maybe some rows full of zeros at the bottom. The unique solution can be read from the rightmost column.
    2. There may be infinitely many solutions. This happens if the system is consistent but at least one of the variables is free.

Example (not in the lecture notes)

1. Solve the following linear system.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array}$$

Step 1: Reduce the augmented matrix to RREF.

$$\begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2: Identify leading variables (x_1, x_2) and free variablse x_3, x_4, and write the general solution.

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (2 + t - 3s, 1 - 2t + 4s, t, s) : \ \ t, s \in \mathbb{R} \\ &= (2, 1, 0, 0) + t(1, -2, 1, 0) + s(-3, 4, 0, 1) : \ \ t, s \in \mathbb{R}. \end{aligned}$$

Example (Part 2)

2. Solve the following linear system.

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & 1 \end{array}$$

We must describe all simultaneous solutions of the first three equations that also satisfy the fourth. A solution of the first three has the form

$$(x_1, x_2, x_3, x_4) = (2 + t - 3s, 1 - 2t + 4s, t, s),$$

for real numbers t and s. Insert this information into the fourth equation:

$$2(2+t-3s)-(1-2t+4s)-3t+4s = 1 \Longrightarrow 3+t-6s = 1 \Longrightarrow t = -2+6s.$$

The parameters t and s are no longer independently free.

$$\begin{array}{rcl} (x_1, x_2, x_3, x_4) & = & (2 + (-2 + 6s) - 3s, 1 - 2(-2 + 6s) + 4s, -2 + 6s, s) \\ & = & (3s, 5 - 8s, -2 + 6s, s) : \ s \in \mathbb{R} \\ & = & (0, 5, -2, 0) + s(3, -8, 6, 1) : \ s \in \mathbb{R}. \end{array}$$

Example (Part 3)

3. Solve the following linear system.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & 1 \\ 3x_1 & - & 2x_2 & - & 2x_3 & - & 5x_4 & = & 10 \end{array}$$

Simultaneous solutions of the first four equations have the form

$$(x_1, x_2, x_3, x_4) = (3s, 5 - 8s, -2 + 6s, s) : \ s \in \mathbb{R}.$$

Check for values of s for which this also satisfies Equation 5:

$$3(3s) - 2(5 - 8s) - 2(-2 + 6s) - 5s = -6 + 8s = 10 \implies 8s = 6, \ s = 2.$$

Unique solution: $(x_1, x_2, x_3, x_4) = (6, -11, 10, 2)$.

Example (Part 4)

3. Show that the following linear system is inconsistent.

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & 1 \\ 3x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & 3 \end{array}$$

Simultaneous solutions of the first four equations have the form

$$(x_1, x_2, x_3, x_4) = (3s, 5 - 8s, -2 + 6s, s) : \; s \in \mathbb{R}.$$

Check for values of s for which this also satisfies Equation 5:

$$3(3s) + 2(5 - 8s) + 2(-2 + 6s) - 5s = 6 \neq 3.$$

No simultaneous solution of the first four equations also satisfies the last one, the system is inconsistent.

Elementary row operations may themselves be interpreted as matrix multiplication exercises.

- We write I_m for the $m \times m$ identity matrix
- We write $E_{i,j}$ for the matrix that has 1 in the (i, j)-position and zeros everywhere else.

Theorem 11

Let A be a $m \times m$ matrix. Then elementary row operations on A amount to multiplying A on the left by $m \times m$ matrices, as follows:

1. *Mutiplying Row i by the non-zero scalar α is equivalent to multiplying A on the left by the matrix $I_m + (\alpha - 1)E_{i,i}$.*
2. *Switching Rows i and k amounts to multiplying A on the left by the matrix $I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,i}$.*
3. *Adding $\alpha\times$ Row i to Row k amounts to multiplying A on the left by the matrix $I_m + \alpha E_{k,i}$.*

Elementary Row Operations as Matrix Multiplication

Matrices of the three types described in Theorem 11 are sometimes referred to as *elementary matrices*. They are always invertible, and their inverses are also elementary matrices. The statement that every matrix can be reduced to RREF through a sequence of EROs is equivalent to saying that for every matrix A with m rows, there exists a $m \times m$ matrix B, which is a product of elementary matrices, with the property that BA is in RREF.

Exercise 12

Write down the inverse of an elementary matrix of each of the three types, and show that it is also an elementary matrix.
(Hint: Think about how to reverse an elementary row operation, with another elementary row operation).

Exercise 13

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Using Gauss-Jordan elimination to calculate matrix inverses

Suppose that $A \in M_n(\mathbb{F})$, for some field $\mathbb{F}$. If A is invertible, let $v_1, v_2, \dots, v_n$ be the columns of its inverse. Then

$$AA^{-1} = A \begin{bmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{bmatrix} = A \begin{bmatrix} | & | & \dots & | \\ Av_1 & Av_2 & \dots & Av_n \\ | & | & \dots & | \end{bmatrix} = I_n.$$

For each i, Av_i is the ith column of the identity matrix, which has 1 in position i and zeros elsewhere. This means that v_i is the solution of the linear system $Av_i = e_i$, where e_i is column i of the identity matrix, and the variables are the unknown entries of v_i.

We need to this for each column, but we can combine this into a single process by writing $e_1, e_2, \dots, e_n$ as n distinct columns in the "right hand side" of a $n \times 2n$ augmented matrix.

Example of inverse calculation

Example Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

To calculate A^{-1}, We apply Gauss-Jordan elimination to the 3×6 matrix below

$$A' = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

We conclude

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}.$$