Chapter 3: Linear transformations, eigenvectors, similarity

Themes for Chapter 3

- It is useful to be able to move between different bases for a given vector space;
- One basis may be far better than another for describing a paricular linear transformation - the standard basis is not always the most useful one;
- Everything can be interpreted in terms of matrix algebra, although the setup takes some work.

How to recognize a basis of $\mathbb{F}^n$

The dimension of the space $\mathbb{F}^n$ is n - the standard basis consists of the column vectors $e_1, \dots, e_n$, where e_i has 1 in position i and zeros in all other positions.
For example, in $\mathbb{R}^3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the standard basis $\mathcal{E} = \{e_1, e_2, e_3\}$.
How can we recognize a basis of $\mathbb{F}^n$? It should have n elements, which should be column vectors in $\mathbb{F}^n$. But some sets of three column vectors in $\mathbb{R}^3$ are bases of $\mathbb{R}^3$ and some are not. How do we know?

Bases of $\mathbb{F}^n$

Theorem Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be any set of n vectors in $\mathbb{F}^n$. Then $\mathcal{B}$ is a basis of $\mathbb{F}^n$ if and only if the matrix A whose columns are $v_1, \dots, v_n$ has an inverse in $M_n(\mathbb{F})$.

$\Longleftarrow$ Suppose that A has an inverse of in $M_n(\mathbb{F})$. Then $AA^{-1} = I_n$, and $Aw_1 = e_1$, where w_1 is the first column of A^{-1}. It follows that e_1 is a linear combination of $v_1, \dots, v_n$. Similarly each e_i is in the linear span of $\{v_1, \dots, v_n\}$, and so $\{v_1, \dots, v_n\}$ is a spanning set of $\mathbb{F}^n$.

$\Longrightarrow$ Suppose that $\mathcal{B}$ is a basis of $\mathbb{F}_n$. Then e_1 is a linear combination of the columns of A, and so $e_1 = Aw_1$, for some $w_1 \in \mathbb{F}^n$. Similarly $e_i = Aw_i$, for $i = 2, \dots, n$. It follows that $AW = I_n$, where W is the matrix in $M_n(\mathbb{F})$ whose columns are $w_1, \dots, w_n$.

Moving between two bases

Suppose we have another basis $\mathcal{B} = \{b_1, b_2, b_3\}$ of R^3 (besides the standard basis), where

$$b_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Question: Suppose we have some vector in $\mathbb{R}^3$, for example $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

What are the coordinates of v with respect to $\mathcal{B}$?

Another Question: Why would we want to know this?

How to write the $\mathcal{B}$-coordinates of v?

If we knew how to write e_1, e_2 and e_3 as a linear combination of b_1, b_2, b_3, we could do the same for v (or any vector).
The $\mathcal{B}$-coordinates of e_1 are the values of x, y, z in the unique solution of

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or } B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = e_1.$$

The corresponding values are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which means they are the entries of Column 1 of B^{-1}. In the same way, the $\mathcal{B}$-coordinates of e_1 and e_3 are given by Columns 2 and 3 of B^{-1}.

The matrices B and B^{-1}

For our example:

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

Looking at (for example) Column 2 of B^{-1} we can confirm that its entries are the $\mathcal{B}$-coordinates of e_2.

The change of basis matrix

Now for the $\mathcal{B}$-coordinates of $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. We write $[v]_{\mathcal{B}}$ for the column whose entries are the $\mathcal{B}$-coordinates of v. We can now achieve this through a matrix-vector product.

$$\begin{aligned} v = 2e_1 + 1e_2 + 3e_3 \Longrightarrow [v]_{\mathcal{B}} &= 2[e_1]_{\mathcal{B}} + 1[e_2]_{\mathcal{B}} + 3[e_3]_{\mathcal{B}} \\ &= \begin{bmatrix} \\ [e_1]_{\mathcal{B}} & [e_2]_{\mathcal{B}} & [e_3]_{\mathcal{B}} \\ \\ \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

Conclusion: $v = 6b_1 + \frac{9}{2}b_2 + \frac{1}{2}b_3$.

Exercise: Confirm this conclusion by direct calculation.

General conclusion

To find the $\mathcal{B}$-coordinates of *any* vector v in $\mathbb{R}^3$, what we need to do is multiply v on the left by the *change of basis* matrix from the standard basis to $\mathcal{B}$. This is the inverse of the matrix whose columns are the elements of $\mathcal{B}$ (written in the standard basis).

Learning outcomes for Section 3.1

1. How to recognize when a set of n column vectors in $\mathbb{R}^n$ (or $\mathbb{F}^n$) forms a basis.
2. To recognize that elements of $\mathbb{R}^n$ (or $\mathbb{F}^n$) have different coordinates with respect to different bases.
3. To use the change of basis matrix to write the coordinates of any vector in $\mathbb{F}^n$ with respect to a given basis.