

Lecture 12: Consequences of the exchange lemma

Theorem 21

If V is a finite dimensional vector space over a field \mathbb{F} , then every basis of V has the same number of elements.

Proof.

Let B_1 and B_2 be bases of V . Then B_1 is linearly independent and B_2 is a spanning set of V , so $|B_1| \leq |B_2|$ by Theorem 20. Also, B_2 is linearly independent and B_1 is a spanning set of V , so $|B_2| \leq |B_1|$ by Theorem 20. Hence $|B_1| = |B_2|$. □

Definition The number of elements in any (hence every) basis of a finite dimensional vector space V is called the *dimension* of V , denoted $\dim V$.

An Example

Let V be the space of skew-symmetric matrices in $M_3(\mathbb{R})$ (a matrix A is *skew-symmetric* if $A^T = -A$). Then

e.g.

$$\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

$$V = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

skew symmetric:
 $a = -a$ $b = -b$
 $c = -c$

The typical element of V noted above can be written as

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}),$$

where E_{ij} is the matrix with 1 in the (i, j) -position and zeros elsewhere.

We see that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a spanning set of V .

This set is also linearly independent. We conclude that

$\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a basis of V and that $\dim V = 3$.

More on Bases and Dimension

Recall (Steinitz Exchange Theorem) In a vector space V , if L is any linearly independent set and S is any finite spanning set, then $|L| \leq |S|$.

Let V be a vector space of dimension n over a field \mathbb{F} .

Lemma 1 Every linearly independent subset of V with n elements is a basis of V .

Lemma 2 Every spanning set of V with n elements is a basis of V .

Lemma 3 If L is a linearly independent subset of V , then L can be extended to a basis of V .

Lemma 1 If $L = \{v_1, \dots, v_n\}$ is linearly independent and is not a spanning set, then there is some $v \in V$ with $v \notin \langle L \rangle$. But then $\{v_1, \dots, v_n, v\}$ would be a linearly independent set with more than n elements.

There is really only one \mathbb{F} -vector space of each dimension!

For any field \mathbb{F} , \mathbb{F}^n denotes the space of all column vectors with n entries.

Suppose that V is a \mathbb{F} -vector space with $\dim V = n$, and let

$B = \{v_1, \dots, v_n\}$ be a basis of V over \mathbb{F} . For every element $v \in V$, there is a unique expression for v as a linear combination of the elements of B :

$$v = \underline{a_1}v_1 + \dots + \underline{a_n}v_n \quad v \longleftrightarrow (a_1, \dots, a_n) \longleftrightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

We refer to $\underline{a_1, \dots, a_n}$ as the **coordinates** of v with respect to the basis B . With this association, we can consider v to be represented by the column vector in \mathbb{F}^n whose entries are a_1, \dots, a_n .

This association defines a bijective correspondence between V and \mathbb{F}^n , and means that we can identify these two vector spaces as being essentially the same.