Theorem 21

If V is a finite dimensional vector space over a field $\mathbb{F}$, then every basis of V has the same number of elements.

Proof.

Let B_1 and B_2 be bases of V. Then B_1 is linearly independent and B_2 is a spanning set of V, so $|B_1| \leq |B_2|$ by Theorem 20. Also, B_2 is linearly independent and B_1 is a spanning set of V, so $|B_2| \leq |B_1|$ by Theorem 20. Hence $|B_1| = |B_2|$. $\square$

Definition The number of elements in any (hence every) basis of a finite dimensional vector space V is called the *dimension* of V, denoted $\dim V$.

An Example

Let V be the space of skew-symmetric matrices in $M_3(\mathbb{R})$ (a matrix A is *skew-symmetric* if $A^T = -A$). Then

$$V = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The typical element of V noted above can be written as

$$\begin{aligned} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} &= a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}), \end{aligned}$$

where E_{ij} is the matrix with 1 in the (i, j)-position and zeros elsewhere. We see that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a spanning set of V. This set is also linearly independent. We conclude that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a basis of V and that $\dim V = 3$.

Recall (Steinitz Exchange Theorem) In a vector space V, if L is any linearly independent set and S is any finite spanning set, then $|L| \leq |S|$.

Let V be a vector space of dimension n over a field $\mathbb{F}$.

Lemma 1 *Every* linearly independent subset of V with n elements is a basis of V.

Lemma 2 *Every* spanning set of V with n elements is a basis of V.

Lemma 3 If L is a linearly independent subset of V, then L can be extended to a basis of V.

There is really only one $\mathbb{F}$-vector space of each dimension!

For any field $\mathbb{F}$, $\mathbb{F}^n$ denotes the space of all column vectors with n entries.

Suppose that V is a $\mathbb{F}$-vector space with $\dim V = n$, and let $B = \{v_1, \dots, v_n\}$ be a basis of V over $\mathbb{F}$. For every element $v \in V$, there is a unique expression for v as a linear combination of the elements of B:

$$v = a_1 v_1 + \cdots + a_n v_n$$

We refer to $a_1, \dots, a_n$ as the coordinates of v with respect to the basis B. With this association, we can consider v to be represented by the column vector in $\mathbb{F}^n$ whose entries are $a_1, \dots, a_n$.

This association defines a bijective correspondence between V and $\mathbb{F}^n$, and means that we can identify these two vector spaces as being essentially the same.

Bases of $\mathbb{F}^n$

The standard basis of $\mathbb{F}^n$ is $\{e_1, \dots, e_n\}$, where e_i has 1 in position i and 0 in all other positions.

Theorem Let $B = \{v_1, \dots, v_n\}$ be any set of n vectors in $\mathbb{F}^n$. Then B is a basis of $\mathbb{F}^n$ if and only if the matrix A whose columns are $v_1, \dots, v_n$ has an inverse in $M_n(\mathbb{F})$.

Proof ($\Longleftarrow$) Suppose that A has an inverse of in $M_n(\mathbb{F})$. Then $AA^{-1} = I_n$, and $Aw_1 = e_1$, where w_1 is the first column of A^{-1}. It follows that e_1 is a linear combination of $v_1, \dots, v_n$.

($\Longrightarrow$) On the other hand, suppose that B is a basis of $\mathbb{F}_n$. Then e_1 is a linear combination of the columns of B, and so $e_1 = Bw_1$, for some $w_1 \in \mathbb{F}^n$. Similarly $e_i = Bw_i$, for $i = 2, \dots, n$. It follows that $AW = I_n$, where W is the matrix in $M_n(\mathbb{F})$ whose columns are $w_1, \dots, w_n$, and hence A has an inverse in $M_n(\mathbb{F})$.