Definition A *basis* of a vector space V is a spanning set of V that is linearly independent. [Plural: bases]

Lemma 19

If S is a finite spanning set of a vector space V, then S contains a basis of V.

Proof.

If S is not linearly independent, then some element v_1 of S is in the span of the other elements of S, and $S_1 := S \setminus \{v_1\}$ is again a spanning set of V. If S_1 is not linearly independent, then we can discard an element of S_1 that is in the linear span of the others, to form a smaller spanning set S_2. Since S is finite, this process cannot continue indefinitely, and it concludes with a linearly independent spanning set of V. □

The number of elements in a basis

We will show that if V has a finite basis, then *every* basis has the same number of elements. This number is then referred to as the *dimension* of V. The key to this is to show that the number of elements in *any* spanning set of V is an upper bound for the number of elements in *any* linearly independent subset of V.

Theorem 20

[Steinitz exchange lemma] Let V be a vector space over a field $\mathbb{F}$, and suppose that $S = \{\underbrace{v_1, \dots, v_n}\}$ is a spanning set of V. Then the number of elements in a linearly independent subset of V cannot exceed n.

Proof Let $S = \{v_1, \dots, v_n\}$ be a *spanning set* of V. Let $\boxed{L = \{y_1, \dots, y_k\}}$ be a linearly independent subset of V. We need to show $\boxed{k \leq n}$.

Note No proper subset of L is a spanning set of V.

Proof We adopt S by replacing elements of S with elements of L one by one, retaining a spanning set at all stages.

Step 1 $y_1 \in \langle S \rangle \Rightarrow \boxed{y_1 = a_1 v_1 + \dots + a_n v_n}$ for scalars a_i

The a_i are not all 0, we can assume $a_1 \neq 0$ (after relabelling if necessary).

Now $a_1 v_1 = y_1 - a_2 v_2 - \dots - a_n v_n \Rightarrow a_1 v_1 \in \langle y_1, v_2, \dots v_n \rangle$

$$\Rightarrow v_1 \in \langle y_1, v_2, \dots, v_n \rangle \quad (\text{since } a_1 \neq 0)$$

So $v_1, v_2, \dots, v_n$ are all in the span of $\underline{S_1 := \{y_1, v_2, \dots v_n\}}$

i.e. S_1 is again a spanning set of V.

Step 2 $y_2 \in \langle S_1 \rangle$. And y_2 is not just a scalar multiple of y_1. We can assume that an expression for y_2 as a combination of elements of S_2 involves v_2 with non-zero coefficient

Then $V_2 = \{y_1, y_2, v_3, \dots, v_n\}$, and $S_2 := \{y_1, y_2, v_3, \dots, v_n\}$ is a spanning set of V.

__Step 3__ Continue in this manner, replacing at each step an element from the original S with the next element of L. After i steps, we have a spanning set S_i containing the first i elements of L and $n-i$ elements from the original S.

__Conclusion__ After $k-1$ steps, we have a spanning set $S_{k-1} = \{y_1, y_2, \dots, y_{k-1}, \dots\}$ of V. Since $y_1, \dots, y_{k-1}$ don't span V (their linear span misses y_k), there are some elements from the original S still present in S_{k-1}. This means $k-1 < n$, i.e. $\boxed{k \leq n}$

Consequences of the exchange lemma

Theorem 21

If V is a finite dimensional vector space over a field $\mathbb{F}$, then every basis of V has the same number of elements.

Proof.

Let B_1 and B_2 be bases of V. Then B_1 is linearly independent and B_2 is a spanning set of V, so $|B_1| \leq |B_2|$ by Theorem 20. Also, B_2 is linearly independent and B_1 is a spanning set of V, so $|B_2| \leq |B_1|$ by Theorem 20. Hence $|B_1| = |B_2|$. $\square$

Definition The number of elements in any (hence every) basis of a finite dimensional vector space V is called the *dimension* of V, denoted $\boxed{\dim V.}$