Lecture 11: Bases and dimension

Definition A *basis* of a vector space V is a spanning set of V that is linearly independent. [Plural: bases]

Lemma 19

If S is a finite spanning set of a vector space V, then S contains a basis of V.

Proof.

If S is not linearly independent, then some element v_1 of S is in the span of the other elements of S, and $S_1 := S \backslash \{v_1\}$ is again a spanning set of V. If S_1 is not linearly independent, then we can discard an element of S_1 that is in the linear span of the others, to form a smaller spanning set S_2. Since S is finite, this process cannot continue indefinitely, and it concludes with a linearly independent spanning set of V. □

The number of elements in a basis

We will show that if V has a finite basis, then *every* basis has the same number of elements. This number is then referred to as the *dimension* of V. The key to this is to show that the number of elements in *any* spanning set of V is an upper bound for the number of elements in *any* linearly independent subset of V.

Theorem 20

[Steinitz exchange lemma] Let V be a vector space over a field $\mathbb{F}$, and suppose that $S = \{v_1, \dots, v_n\}$ is a spanning set of V. Then the number of elements in a linearly independent subset of V cannot exceed n.

Proof Let $S = \{v_1, \dots, v_t\}$ be a *spanning set* of V. Let $L = \{y_1, \dots, y_k\}$ be a linearly independent subset of V. We need to show $k \leq t$.

Theorem 21

If V is a finite dimensional vector space over a field $\mathbb{F}$, then every basis of V has the same number of elements.

Proof.

Let B_1 and B_2 be bases of V. Then B_1 is linearly independent and B_2 is a spanning set of V, so $|B_1| \leq |B_2|$ by Theorem 20. Also, B_2 is linearly independent and B_1 is a spanning set of V, so $|B_2| \leq |B_1|$ by Theorem 20. Hence $|B_1| = |B_2|$. □

Definition The number of elements in any (hence every) basis of a finite dimensional vector space V is called the *dimension* of V, denoted $\dim V$.

Let V be the space of skew-symmetric matrices in $M_3(\mathbb{R})$ (a matrix A is *skew-symmetric* if $A^T = -A$). Then

$$V = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The typical element of V noted above can be written as

$$\begin{aligned} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} &= a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}), \end{aligned}$$

where E_{ij} is the matrix with 1 in the (i, j)-position and zeros elsewhere. We see that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a spanning set of V. This set is also linearly independent. We conclude that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a basis of V and that $\dim V = 3$.