Lecture 10: An example in $\mathbb{R}^2$

The set $S = \left\{ \binom{3}{1}, \binom{2}{1}, \binom{1}{-1} \right\}$ is a spanning set of the vector space $\mathbb{R}^2$ of all real column vectors with two entries. If $v = \binom{a}{b} \in \mathbb{R}^2$, we can write v as a linear combination of the elements of S, for example by writing

e.g. $\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 8\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 8\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a+b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a-b)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - b\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This is not the only way to do it. We could also write

$$\begin{pmatrix} a \\ b \end{pmatrix} = (4a+b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-5a-b)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 20\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 24\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 8\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

We could forget about the third element of S and just write

$$\boxed{\begin{pmatrix} a \\ b \end{pmatrix}} = (a-2b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a+3b)\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$\begin{pmatrix} 4 \\ 4 \end{pmatrix} = -4\begin{pmatrix} 3 \\ 1 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So all three elements of S are not needed to span $\mathbb{R}^2$. We could do it just with the subset $\boxed{\left\{ \binom{2}{1}, \binom{3}{1} \right\}}$. Note that $\binom{1}{-1}$ is a $\mathbb{R}$-linear combination of the other two elements of S. If we drop this element from S, we can still recover it in the span of the remaining elements.

Finite dimensional and infinite dimensional spaces

Lemma 16

Suppose that $S_1 \subset S$, where S is a subset of a vector space V. Then $\langle S_1 \rangle \subseteq \langle S \rangle$, and $\langle S_1 \rangle = \langle S \rangle$ if and only if every element of $S \backslash S_1$ is a linear combination of elements of S_1.

We finish this section by noting the distinction between a *finite dimensional* and *infinite dimensional* vector space.

Definition 17

A vector space is said to be *finite dimensional* if it has a finite spanning set. A vector space that does not have a finite spanning set is *infinite dimensional*.

Two examples of infinite dimensional vector spaces

1. The vector space $\mathbb{R}[x]$ of all polynomials with real coefficients is infinite dimensional. To see this, let S be any finite subset of $\mathbb{R}[x]$ (i.e. a finite set of polynomials). Let x^k be the highest power of x to appear in any element of S. Then no linear combination of elements of S has degree exceeding k, so the linear span of S cannot be all of $\mathbb{R}[x]$.
2. The set $\mathbb{R}$ of real numbers is infinite dimensional as a vector space over the field $\mathbb{Q}$ of rational numbers.

Definition Let S be a subset of a vector space V, having at least 2 elements. Then S is *linearly independent* if no element of S is a linear combination of the other elements of S (equivalently, if no element of S belongs to the span of the other elements of S).
A subset consisting of a single element is linear independent, provided that its unique element is not the zero vector.

To decide if a given set is linearly independent, the above definition is maybe not the most useful formulation, because it requires us to check something separately for each element of S, which could take a lot of work. The following altenative version is often more useful in practice.

Definition (Equivalent version) Let S be a non-empty subset of a vector space V. Then S is *linearly independent* if the only way to write the zero vector in V as a linear combination of elements of S is to take all the coefficients to be 0.

Equivalence of the two definitions

Let $S = \{v_1, \dots, v_k\}$ and suppose that $v_1 \in \langle v_2, \dots, v_k \rangle$. Then

S is not independent according to the first definition

$$\boxed{v_1 = a_2 v_2 + \cdots + a_k v_k,}$$

and

$$0_V = -v_1 + a_2 v_2 + \cdots + a_k v_k$$

is an expression for the zero vector as a linear combination of elements of S, whose coefficients are not all zero.

On the other hand, suppose that

$$\boxed{0_V = c_1 v_1 + \cdots + c_k v_k}$$

where the scalars c_i are not all zero. If $c_1 \neq 0$ (for example), then the above equation can be rearranged to express v_1 as a linear combination of $v_2, \dots, v_k$:

Scalars in $\mathbb{F}$.

$$v_1 = -\frac{c_2}{c_1} v_2 - \cdots - \frac{c_k}{c_1} v_k$$

In $\mathbb{R}^3$, let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} \right\}$.

To determine whether S is linearly independent, we must investigate whether the system of equations

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 2y - 3z = 0$
$2x + 3y + 8z = 0$
$-x + 2y + 3z = 0$

has solutions other than $\boxed{(x, y, z) = (0, 0, 0)}$. The augmented matrix of this system, and its RREF, are

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 2 & 3 & 8 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x & y & z & \end{array} \end{array}$$

$x + z = 0$
$y + 2z = 0$

An example in $\mathbb{R}^3$

Thus for any t, $(x, y, z) = (-t, -2t, t)$ is a solution, and for example by taking $t = 1$ we see that

$$\boxed{-1\begin{bmatrix}1\\2\\-1\end{bmatrix} - 2\begin{bmatrix}-2\\3\\2\end{bmatrix} + 1\begin{bmatrix}-3\\8\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}},$$

and hence that each of the three elements of S is a linear combination of the other two. So S is not linearly independent (we say that S is *linearly dependent*).

Characterizations of linearly independent sets

Let S be a subset of a vector space V.

1. S is linearly independent if S is a *minimal* spanning set of its linear span - no proper subset of S spans the same subspace of V that S does, or every proper sub~~space~~ set of S spans a strictly smaller subspace than S itself.
2. S is linearly independent if every element of $\boxed{\langle S \rangle}$ has a *unique* expression as a linear combination of elements of S.
3. Another version of 2. above: S is linearly independent if every element of the span of S has *unique coordinates* in terms of the elements of S. $S = \{v_1, \dots, v_k\}$ $v = a_1 v_1 + \dots + a_k v_k$ $a_i \in \mathbb{F}$ Choice of a_i is unique

So a linearly independent set in a vector space V is a *minimal* or *irredundant* spanning set for its linear span. If its linear span happens to be all of V, it gets a special name.

Definition 18

A *basis* of a vector space V is a spanning set of V that is linearly independent. (plural: bases)