

Chapter 2

The Real Numbers

2.1 The set \mathbb{R} of real numbers

This section involves a consideration of properties of the set \mathbb{R} of real numbers, the set \mathbb{Q} of rational numbers, the set \mathbb{Z} of integers and other related sets of numbers. In particular, we will be interested in what is special about \mathbb{R} , what distinguishes the real numbers from the rational numbers and why the set of real numbers is such an interesting and important thing that there is a whole branch of mathematics (real analysis) devoted to its study.

WHAT IS \mathbb{R} ?

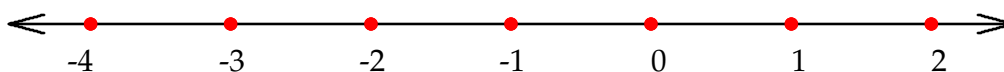
There are at least two useful ways to think about what real numbers are, and before considering them it is useful to first recall what *integers* are and what *rational numbers* are.

Integers are “whole numbers”. The set of integers is denoted by \mathbb{Z} :

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The notation “ \mathbb{Z} ” comes from the German word *Zahlen* (numbers).

On the number line, the integers appear as an infinite set of evenly spaced points. The integers are exactly those numbers whose decimal representations have all zeroes after the decimal point.



Note that the integers on the number line are separated by *gaps*. For example there are no integers in the chunk of the number line between $\frac{7}{5}$ and $\frac{63}{32}$.

The set of integers is *well-ordered*. This means (more or less) that given any integer, it makes sense to talk about *the next* integer after that one. For example, the next integer after 3 is 4. To see why this property (which might not seem very remarkable at first glance) is something worth bothering about and to understand what it says, it might be helpful to observe that the same property does not hold for the set \mathbb{Q} of *rational numbers* described below.

A *rational number* is a number that can be expressed as a fraction with an integer as the numerator and a non-zero integer as the denominator. The set of all rational numbers is denoted by \mathbb{Q} (\mathbb{Q} is for *quotient*).

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}.$$

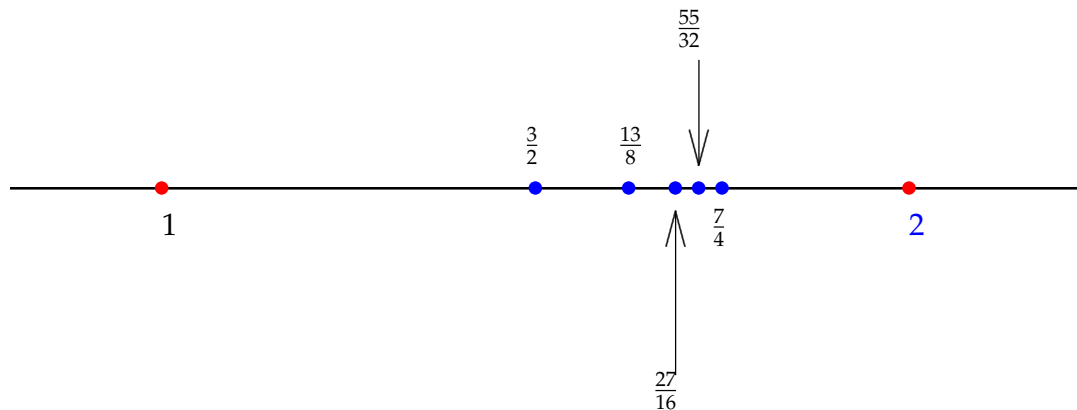
Note: The above statement can be read as “ \mathbb{Q} is the set of all numbers that can be written in the form $\frac{a}{b}$ where a is an integer, b is an integer, and b is not zero”. In order to be able to make sense of written mathematics it is essential to be able to read all the parts of statements of this kind and form a clear mental impression of what is being said. In written mathematics, every mark on the page has

meaning and you are expected to attend to that. This takes practice and care and it is not optional if you want to learn mathematics. Mathematical writing is and needs to be entirely unambiguous - this means it has to be technical and fussy, but there is no opportunity for misinterpretation once you are familiar with the relevant definitions and notation.

So \mathbb{Q} includes such numbers as $\frac{5}{7}$, $\frac{-8}{16}$, $\frac{3141}{22445}$ and so on. It includes all the integers, since any integer n can be written in the form of a fraction as $\frac{n}{1}$. The rational numbers are exactly those numbers whose decimal representations either terminate (i.e. all digits are 0 from some point onwards) or repeat (i.e. from some point onwards the decimal part consists of some string of digits repeated over and over without end).

Note: The statement that \mathbb{Q} includes all the integers can be written very concisely as $\mathbb{Z} \subset \mathbb{Q}$. This says : \mathbb{Z} is a subset of \mathbb{Q} , i.e. every element of \mathbb{Z} is an element of \mathbb{Q} , i.e. every integer is a rational number.

Since rational numbers (by definition) can be written as quotients (or fractions) involving integers, the sets \mathbb{Q} and \mathbb{Z} are closely related. However, on the number line these sets do not resemble each other at all. As mentioned above, the integers are “spaced out” on the number line and there are gaps between them. By contrast, the rational numbers are “densely packed” in the number line. The diagram below is intended to show that the stretch of the number line between 1 and 2 contains infinitely many rational numbers - we can’t draw infinitely many of them in a picture, but hopefully this picture indicates how we can keep adding more and more of them indefinitely. By contrast with the situation for \mathbb{Z} , there are no stretches of the number line that are without rational numbers.



Note: Does this picture persuade you that there are infinitely many rational numbers between 1 and 2? If not, do you believe this statement? It is up to you to consider its plausibility and reach a conclusion.

Related to these remarks is the observation that the set of rational numbers is not *well-ordered*. Given a particular rational number, there is no *next* rational number after it. For example 0 is a rational number, but there is no *next rational number* after 0 on the number line. This is the same as saying that there is no *smallest* positive rational number. If we had a candidate for this title, we could divide by 2 and we would have a smaller but still positive rational number.

Exercise 2.1.1. Choose a stretch of the number line - for example the stretch from $-\frac{7}{4}$ to $-\frac{11}{8}$ (but pick your own example). Persuade yourself that your stretch contains infinitely many rational numbers.

So the rational numbers are not sparse like the integers. They come close to covering the whole number line in the sense that any stretch (however short) of the number line contains infinitely many rational numbers. The idea of visualizing the points corresponding to rational numbers as a “mist” on the number line has been suggested.

Now imagine an infinite straight line, on which the integers are marked (in order) by an infinite set of evenly spaced dots. Imagine that the rational numbers have also been marked by dots, so that the dot representing $\frac{3}{2}$ is halfway between the dot representing 1 and the dot representing 2, and so on. (Of course it is not physically possible to do all this marking, but it's possible to imagine what the picture would look like). At this stage a lot of dots have been marked - every stretch of the line, no matter how short, contains an infinite number of marked dots.

However, many points on the line remain unmarked. For example, somewhere between the dot representing the rational number 1.4142 and the dot representing the rational number 1.4143 is an unmarked point that represents the real number $\sqrt{2}$. This number is not rational - it cannot be expressed in the form $\frac{a}{b}$ for integers a and b . Somewhere between the dot that marks 3.1415 and the dot that marks 3.1416 is an unmarked point representing the irrational number π . The set \mathbb{R} of *real numbers* is the set of numbers corresponding to *all* points on the line, marked or not. It includes both the rational and irrational numbers.

Note: Because the examples of irrational numbers that are usually cited are things like $\sqrt{2}$, π and e , you could get the impression that irrational numbers are special and rare. This is far from being true. In a very precise way that we will see later, the irrational numbers are more numerous than the rational numbers. If you think of the points representing irrational numbers as a "mist" on the number line, it would be a denser mist than the one for rational numbers. If all the rational numbers were coloured blue on the number line and all the irrational numbers were coloured red, the whole number line would be a jumble of blue and red points, but there would be more red than blue. If you zoomed in, as far as you like, on any section of the number line, however short, both blue and red would still appear, and there would still be more red than blue.

Exercise 2.1.2. Write down five irrational numbers between 4 and 5.

Hint: If you don't know what to do, start with $\sqrt{2}$ or some other number that you know is irrational. The number $\sqrt{2}$ is not between 4 and 5 obviously. What adjustments can be made?

Exercise 2.1.3. Suppose that a is a rational number and b is an irrational number.

- Might $a + b$ be irrational?
- Must $a + b$ be irrational?
- Might ab be irrational?
- Must ab be irrational?
- Might the product of two irrational numbers be irrational?
- Must the product of two irrational numbers be irrational?

Hint : If in doubt, give yourself some examples and investigate.

To conclude this section we propose two different ways of thinking about the set of real numbers.

1. (*Arithmetic description*) The set \mathbb{R} of real numbers consists of *all* numbers that can be written as (possibly non-terminating and possibly non-repeating) decimals.

This description is accurate and conceptually it is valuable, but it is not of much practical use because it is not possible to write out a non-terminating non-repeating decimal or do calculations with it. In reality, when we do calculations with decimals (either by hand or by machine), we truncate them at some point and work with approximations which are rational numbers. The set of numbers that a calculator works with is not the set of real numbers or even the set of rational numbers - it is some subset of \mathbb{Q} that depends on the precision of the instrument.

This arithmetic description of the real numbers highlights the following point. *All* numbers that can be expressed as decimals means *all* numbers that can be written as sequences of the

digits $0, 1, \dots, 9$ (with a decimal point somewhere) with no pattern of repetition necessary in the digits. In the universe of all such things, the ones that terminate (i.e. end in an infinite string of zeroes) or have a repeating pattern from some point onwards are special and rare. These are the rational numbers. The ones that have all zeroes after the decimal point are even more special - these are the integers.

Later we will look at the following questions, which might seem at first glance not to even make sense, but which have interesting and surprising answers.

- Are there more rational numbers than integers?
- Are there more real numbers than rational numbers?

2. (*Geometric description*) The set \mathbb{R} of real numbers is the set of *all* points on the number line. This is a *continuum* - there are no gaps in the real numbers and no point on the line that doesn't correspond to a real number.

Note: As this course proceeds you will need to know what the symbols \mathbb{Z} , \mathbb{Q} and \mathbb{R} mean and be able to recall this information easily. You'll need to be familiar with all the notation involving sets etc. that is used in this section and to be able to use it in an accurate way.

2.2 Subsets of \mathbb{R}

In this section we consider the notions of *finite* and *infinite* sets, and the *cardinality* of a set. Reasonable goals for this section are to become familiar with these ideas and to practice interpreting descriptions of sets that are presented in terse mathematical notation (this means, amongst other things, distinguishing between different kinds of brackets : {}, [], (), etc.).

Definition 2.2.1. A set is finite if it is possible to list its distinct elements one by one, and this list comes to an end.

A set is infinite if any attempt at listing its distinct elements continues indefinitely.

Example 2.2.2. The point of this example is not only to show some finite and infinite sets but also to consider the notation that is used to describe them.

1. $\{1, 2, 3, 4, 5\}$ is a *finite* set - its only elements are the integers 1,2,3,4,5, there are five of them.
2. The interval $[1, 3]$ is an *infinite* set - it consists of *all* the real numbers that are at least equal to 1 and at most equal to 3.

$$[1, 3] := \{x \in \mathbb{R} : 1 \leq x \leq 3\}.$$

Note: The symbol “:=” here means this is a statement of the *definition* of $[1, 3]$.

3. \mathbb{Z} and \mathbb{Q} are *infinite* sets.
4. The set of real solutions of the equation

$$x^5 + 2x^4 - x^2 + x + 17 = 0$$

is a *finite* set. We don't know how many elements it has, but it has at most five, since each one corresponds to a factor of degree 1 of this polynomial of degree 5.

5. The set of prime numbers is *infinite*.
A pair of *twin primes* is a pair of primes that differ by 2 : e.g. 3 and 5, 11 and 13, 59 and 61.
It is not known whether the set of pairs of twin primes is finite or infinite.

Definition 2.2.3. The cardinality of a finite set S , denoted $|S|$, is the number of elements in S .

Example 2.2.4.

1. If $S = \{5, 7, 8\}$ then $|S| = 3$.
2. $|\{4, 10, \pi\}| = 3$
3. $|\{x \in \mathbb{Z} : \pi < x < 3\pi\}| = 6$.
Note: $\{x \in \mathbb{Z} : \pi < x < 3\pi\} = \{4, 5, 6, 7, 8, 9\}$.
4. The cardinality of \mathbb{Q} is infinite.

Remarks

1. The notation “ $|\cdot|$ ” is severely overused in mathematics. This can be a bit annoying since mathematical text is supposed to be entirely unambiguous. If x is a real number, $|x|$ means the absolute value of x . If S is a set, $|S|$ means the cardinality of S . If A is a matrix $|A|$ means the determinant of A . There are other usages as well. It is supposed to be clear from the context what is meant.
2. Defining the concept of cardinality for infinite sets is trickier, since you can't say how many elements they have. We will be able to say though what it means for two infinite sets to have the same (or different) cardinalities.

Example 2.2.5. (A silly example) In a hotel, keys for all the guest rooms are kept on hooks behind the reception desk. If a room is occupied, the key is missing from its hook because the guests have it. If the receptionist wants to know how many rooms are occupied, s/he doesn't have to visit all the rooms to check - s/he can just count the number of hooks whose keys are missing.

There is nothing deep about this example, but it illustrates a point that is important in mathematics. In the example, the occupied rooms are in *one-to-one correspondence* with the empty hooks. This means that each occupied room corresponds to *one and only one* empty hook, and each empty hook corresponds to *one and only one* occupied room. So the number of empty hooks is the same as the number of occupied rooms and we can count one by counting the other.

Definition 2.2.6. Suppose that A and B are sets. Then a *one-to-one correspondence* or a *bijective correspondence* between A and B is a pairing of each element of A with an element of B , in such a way that every element of B is matched to exactly one element of A .

Definition 2.2.7. Suppose that A and B are sets. A function $f : A \rightarrow B$ is called a *bijection* if

- Whenever a_1 and a_2 are different elements of A , $f(a_1)$ and $f(a_2)$ are different elements of B .
- Every element b of B is the image of some element a of A .

Note: Definitions 2.2.6 and 2.2.7 are not really much different from each other, but the second one has a bit more technical machinery of a sort that is sometimes useful in trying to describe correspondences. The example about the keys and rooms is a case of both. The sets A and B here are the respectively the set of empty hooks and occupied rooms. The bijective correspondence is the matching of each empty hook to the room opened by its key, and the "function" f associates to each hook the corresponding room. The fact that different hooks have different images under f says that each key opens only one room, and the fact that every element of B is the image of *some* element of A says that every occupied room is opened by *some* key belonging to an empty hook.

The translation between the concrete context of Example 2.2.5 and the formal definition 2.2.7 is completely unnecessary in terms of understanding the example, but the point is that sometimes the objects we are dealing with don't have a concrete context like this and the formal language is actually necessary to describe the situation. The point of the example is just to show that Definition 2.2.7 is not as obscure or as complicated as it might seem at first glance.

Quite often, in order to determine the cardinality of a set, it is easiest to determine the cardinality of another set with which we know it is in bijective correspondence.

Example 2.2.8. How many integers between 1 and 1000 are perfect squares?

Solution: The list of perfect squares in our range begins as follows

$$1, 4, 9, 16, \dots$$

One way of solving the problem would be to keep writing out successive terms of this sequence until we hit one that exceeds 1000, and then delete that one and count the terms that we have. This is actually more work than we are asked to do, since we are not asked for the list of squares but just the number of them.

Alternatively, we could notice that $(31)^2 = 961$ and $(32)^2 = 1024$. So the numbers $1^2, 2^2, \dots, (31)^2$ are all in the range 1 to 1000 and these are the only perfect squares in that range, the answer to our question is 31.

What we are using here, technically, is the fact that the set of perfect squares in the range of interest is in *bijective correspondence* with the set $\{1, 2, 3, \dots, 31\}$ - the numbers in question are the squares of the first 31 natural numbers. To get the answer 31, it's not really the squares in the range 1 to 1000 that we are counting but the integers in the range 1 to 31.

The following example shows that it could be possible to know that there is a bijective correspondence between two finite sets, without knowing the cardinality of either of them. While this example is a bit contrived, the point of it is to see that sometimes we can show that two sets must be in bijective correspondence even if we don't know much about their elements. This can be a useful device.

Example 2.2.9. Show that the equations

$$x^3 + 2x + 4 = 0 \text{ and } x^3 + 3x^2 + 5x + 7 = 0$$

have the same number of real solutions.

Solution: One way of doing this without having to solve the equations is to demonstrate a bijective correspondence their sets of real equations. We can write

$$\begin{aligned} x^3 + 3x^2 + 5x + 7 &= (x^3 + 3x^2 + 3x + 1) + 2x + 6 \\ &= (x + 1)^3 + (2x + 2) + 4 \\ &= (x + 1)^3 + 2(x + 1) + 4. \end{aligned}$$

This means that a real number a is a solution of the second equation if and only if

$$(a + 1)^3 + 2(a + 1) + 4 = 0$$

i.e. if and only if $a + 1$ is a solution of the first equation.

The correspondence $a \longleftrightarrow a + 1$ is a bijective correspondence between the solution sets of the two equations. So they have the same number of real solutions.

Note: This number is either 1 or 3. Why?

2.3 Infinite sets and cardinality

Recall from the last section that

- The cardinality of a finite set is defined as the number of elements in it.
- Two sets A and B have *the same cardinality* if (and only if) it is possible to match each element of A to an element of B in such a way that every element of each set has exactly one “partner” in the other set. Such a matching is called a *bijjective correspondence* or *one-to-one correspondence*. A bijective correspondence between A and B may be expressed as a *function* from A to B that assigns different elements of B to all the elements of A and “uses” all the elements of B . A function that has these properties is called a *bijection*.

In the case of finite sets, the second point above might seem to be overcomplicating the issue, since we can tell if two finite sets have the same cardinality by just counting their elements and noting that they have the same number. The notion of bijective correspondence is emphasized for two reasons. First, as we saw in Example 2.2.9, it is occasionally possible to establish that two finite sets are in bijective correspondence without knowing the cardinality of either of them. More importantly, we would like to develop some notion of cardinality for infinite sets as well. We can’t count the number of elements in an infinite set. However, for a given pair of infinite sets, we could possibly show that it is or isn’t possible to construct a bijective correspondence between them.

Definition 2.3.1. *Suppose that A and B are sets (finite or infinite). We say that A and B have the same cardinality (written $|A| = |B|$) if a bijective correspondence exists between A and B .*

In other words, A and B have the *same cardinality* if it’s possible to match each element of A to a different element of B in such a way that every element of both sets is matched exactly once. In order to say that A and B have *different* cardinalities we need to establish that it’s impossible to match up their elements with a bijective correspondence. If A and B are infinite sets, showing that such a thing is *impossible* is a formidable challenge.

The remainder of this section consists of a collection of examples of pairs of sets that have the same cardinality. Recall the following definition.

Definition 2.3.2. *The set \mathbb{N} of natural numbers (“counting numbers”) consists of all the positive integers.*

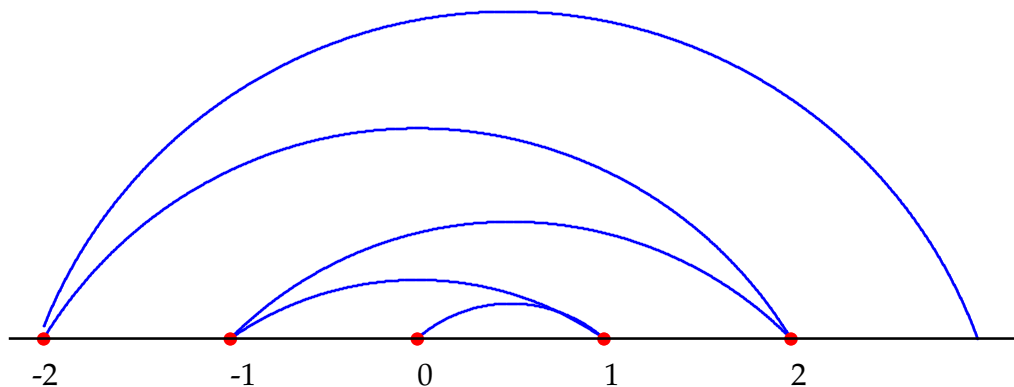
$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

Example 2.3.3. *Show that \mathbb{N} and \mathbb{Z} have the same cardinality.*

The sets \mathbb{N} and \mathbb{Z} are both infinite obviously. In order to show that \mathbb{Z} has the same cardinality of \mathbb{N} , we need to show that the right-hand column of the table below can be filled in with the integers *in some order*, in such a way that each integer appears there exactly once.

\mathbb{N}		\mathbb{Z}
1	\longleftrightarrow	?
2	\longleftrightarrow	?
3	\longleftrightarrow	?
4	\longleftrightarrow	?
\vdots	\longleftrightarrow	\vdots

So we need to list all the integers on the right hand side, in such a way that every integer appears once. Just following the natural order on the integers won’t work, because then there is no first entry for our list. Starting at a particular integer like 0 and then following the natural order won’t work, because then we will never get (for example) any negative integers in our list. Something that *will* work is suggested by following the arcs, starting from 0, in the picture below.



We can start with 0, then list 1 and then -1 , then 2 and then -2 , then 3 and then -3 and so on. This is a systematic way of writing out the integers, in the sense that given any integer, we can identify the one position where it will appear in our list. For example the integer 10 will be Item 20 in our list, the integer -11 will be Item 22.

Our table becomes

\mathbb{N}	\longleftrightarrow	\mathbb{Z}
1	\longleftrightarrow	0
2	\longleftrightarrow	1
3	\longleftrightarrow	-1
4	\longleftrightarrow	2
5	\longleftrightarrow	-2
6	\longleftrightarrow	3
\vdots	\longleftrightarrow	\vdots

If we want to be fully explicit about how this bijective correspondence works, we can even give a formula for the integer that is matched to each natural number. The correspondence above describes a bijective function $f : \mathbb{N} \rightarrow \mathbb{Z}$ given for $n \in \mathbb{N}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

Exercise 2.3.4. What integer occurs in position 50 in our list? In what position does the integer -65 appear?

As well as understanding this example at the informal/intuitive level suggested by the picture above, think about the formula above, and satisfy yourself that it does indeed describe a bijection between \mathbb{N} and \mathbb{Z} . If you are convinced that the two questions above (and all others like them) have unique answers that can be worked out, this basically says that our correspondence between \mathbb{N} and \mathbb{Z} is bijective.

Example 2.3.3 demonstrates a curious thing that can happen when considering cardinalities of infinite sets. The set \mathbb{N} of natural numbers is a *proper subset* of the the set \mathbb{Z} of integers (this means that every natural number is an integer, but the natural numbers do not account for all the integers). Yet we have just shown that \mathbb{N} and \mathbb{Z} are in bijective correspondence. So it is possible for an infinite set to be in bijective correspondence with a proper subset of itself, and hence to have the same cardinality as a proper subset of itself. This can't happen for finite sets (why?).

Putting an infinite set in bijective correspondence with \mathbb{N} amounts to providing a robust and unambiguous scheme or instruction for listing all its elements starting with a first, then a second, third, etc., in such a way that it can be seen that every element of the set will appear exactly once in the list.

Definition 2.3.5. A set is called *countably infinite* (or *denumerable*) if it can be put in bijective correspondence with the set of natural numbers. A set is called *countable* if it is either finite or countably infinite.

Basically, an infinite set is countable if its elements can be listed in an inclusive and organised way. "Listable" might be a better word, but it is not really used. Example 2.3.3 shows that the set \mathbb{Z} of integers is countable. To fully appreciate the notion of countability, it is helpful to look at an example of an infinite set that is not countable. This is coming up in Section 1.4.

Thus according to Definition 2.3.1, the sets \mathbb{N} and \mathbb{Z} have the same cardinality. Maybe this is not so surprising, because \mathbb{N} and \mathbb{Z} have a strong geometric resemblance as sets of points on the number line. What is more surprising is that \mathbb{N} (and hence \mathbb{Z}) has the same cardinality as the set \mathbb{Q} of all rational numbers. These sets do not resemble each other much in a geometric sense. The natural numbers are sparse and evenly spaced, whereas the rational numbers are densely packed into the number line. Nevertheless, as the following construction shows, \mathbb{Q} is a countable set.

Example 2.3.6. *Show that \mathbb{Q} is countable.*

We need to show that the rational numbers can be organized into a numbered list in a systematic way that includes all of them. Such a list is a one-to-correspondence with the set \mathbb{N} of natural numbers. To construct such a list, start with the following array of fractions.

$$\begin{array}{cccccccc}
 \dots & -3/1 & -2/1 & -1/1 & 0/1 & 1/1 & 2/1 & 3/1 & \dots \\
 \dots & -3/2 & -2/2 & -1/2 & 0/2 & 1/2 & 2/2 & 3/2 & \dots \\
 \dots & -3/3 & -2/3 & -1/3 & 0/3 & 1/3 & 2/3 & 3/3 & \dots \\
 \dots & -3/4 & -2/4 & -1/4 & 0/4 & 1/4 & 2/4 & 3/4 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 \end{array}$$

In these fractions, the numerators increase through all the integers as we travel along the rows, and the denominators increase through all the natural numbers as we travel downwards through the columns. Every rational number occurs somewhere in the array. In order to construct a bijective correspondence between \mathbb{N} and the set of fractions in our array, we construct a systematic path that will visit every fraction in the array exactly once. One way of doing this (not the only way) is suggested by the arrows in the following diagram.

$$\begin{array}{cccccccc}
 \dots & -3/1 & -2/1 & \leftarrow -1/1 & 0/1 & \rightarrow 1/1 & 2/1 & \rightarrow 3/1 & \dots \\
 & \uparrow & \downarrow & & \uparrow & & \downarrow & \uparrow & \downarrow \\
 \dots & -3/2 & -2/2 & & -1/2 & \leftarrow 0/2 & \leftarrow 1/2 & 2/2 & 3/2 & \dots \\
 & \uparrow & \downarrow & & \uparrow & & \downarrow & \uparrow & \downarrow \\
 \dots & -3/3 & -2/3 & \rightarrow -1/3 & \rightarrow 0/3 & \rightarrow 1/3 & \rightarrow 2/3 & 3/3 & \dots \\
 & \uparrow & & & \uparrow & & \downarrow & & \\
 \dots & -3/4 & \leftarrow -2/4 & \leftarrow -1/4 & \leftarrow 0/4 & \leftarrow 1/4 & \leftarrow 2/4 & \leftarrow 3/4 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 \end{array}$$

This path determines a listing of all the fractions in the array, that starts as follows

$0/1, 1/1, 1/2, 0/2, -1/2, -1/1, -2/1, -2/2, -2/3, -1/3, 0/3, 1/3, 2/3, 2/2, 2/1, 3/1, 3/2, 3/3, 3/4, \dots$

What this example demonstrates is a bijective correspondence between the set \mathbb{N} of natural numbers and the set of all fractions in our array. A bijective correspondence between some infinite set and \mathbb{N} is really just an ordered listing of the elements of that set ("ordered" here just means for the purpose of the list, and the order in which elements are listed doesn't need to relate to any natural order on the set). This is not (exactly) a bijective correspondence between \mathbb{N} and \mathbb{Q} .

Exercise 2.3.7. *Why not? (Think about this before reading on.)*

The reason why not is that every rational number appears many times in our array. Already in the section of the list above we have $1/1$, $2/2$ and $3/3$ appearing - these represent different entries in our array but they all represent the same rational number. Equally, the fraction $3/4$ appears in the segment of the list above, but $6/8$, $9/12$ and $90/120$ will all appear later.

In order to get a bijective correspondence between \mathbb{N} and \mathbb{Q} , construct a list of all the rational numbers from the array as above, but whenever a rational number is encountered that has already appeared, leave it out. Our list will begin

$0/1, 1/1, 1/2, -1/2, -1/1, -2/1, -2/3, -1/3, 1/3, 2/3, 2/1, 3/1, 3/2, 3/4, \dots$

We conclude that the rational numbers are countable.

Note : Unlike our one-to-one correspondence between \mathbb{N} and \mathbb{Z} in Example 2.3.3, in this case we cannot write down a simple formula to tell us what rational number will be Item 34 on our list (i.e. corresponds to the natural number 34) or where in our list the rational number $292/53$ will appear.

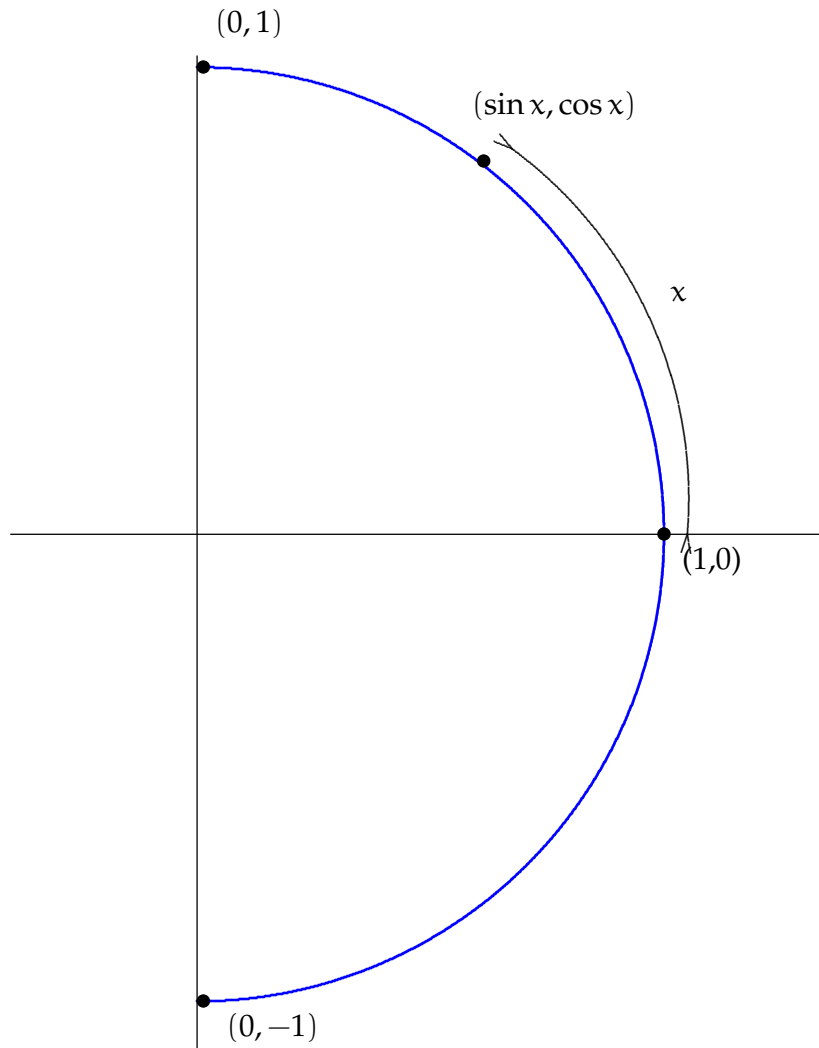
In our next example we show that the set of all the real numbers has the same cardinality as an open interval on the real line.

Example 2.3.8. Show that \mathbb{R} has the same cardinality as the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

Note: $(-\frac{\pi}{2}, \frac{\pi}{2}) = \{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$. The open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ consists of all those real numbers that are strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, not including $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ themselves.

In order to do this we have to establish a bijective correspondence between the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and the full set of real numbers.

An example of a function that provides us with such a bijective correspondence is familiar from calculus/trigonometry. Recall that for a number x in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ $\tan x$ is defined as follows : travel from $(1, 0)$ a distance $|x|$ along the circumference of the unit circle, anti-clockwise if x is positive and clockwise if x is negative. We arrive at a point which is in the right-hand side of the unit circle, because we have travelled a distance of less than $\frac{\pi}{2}$ which would be one-quarter of a full lap. Now $\tan x$ is the ratio of the y -coordinate of this point to the x -coordinate (which are $\sin x$ and $\cos x$ respectively).



Now $\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$, and as x increases from 0 towards $\frac{\pi}{2}$, $\tan x$ is increasing, since $\sin x$ (the y -coordinate of a point on the circle) is increasing and $\cos x$ (the x -coordinate) is decreasing. Moreover, since $\sin x$ is approaching 1 and $\cos x$ is approaching 0 as x approaches $\frac{\pi}{2}$, $\tan x$ is increasing without limit as x approaches $\frac{\pi}{2}$. Thus the values of $\tan x$ run through *all* the positive real numbers as x increases from 0 to $\frac{\pi}{2}$.

For the same reason, the values of $\tan x$ include *every* negative real number exactly once as x runs between 0 and $-\frac{\pi}{2}$.

Thus for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the correspondence

$$x \longleftrightarrow \tan x$$

establishes a bijection between the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and the full set of real numbers. We conclude that the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ has the same cardinality as \mathbb{R} .

Notes:

1. We don't know yet if \mathbb{R} (or $(-\frac{\pi}{2}, \frac{\pi}{2})$) has the same cardinality as \mathbb{N} - we don't know if \mathbb{R} is countable.
2. The interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ may seem like an odd choice for an example like this. A reason for using it in this example is that is convenient for using the \tan function to exhibit a bijection

with \mathbb{R} . However, note that the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is in bijective correspondence with the interval $(-1, 1)$, via the function that just multiplies everything by $\frac{2}{\pi}$.

$$\begin{aligned} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) &\longleftrightarrow (-1, 1) \\ x &\longleftrightarrow \frac{2x}{\pi}. \end{aligned}$$

Exercise 2.3.9. Show that the the open interval $(0, 1)$ has the same cardinality as

1. The open interval $(-1, 1)$
2. The open interval $(1, 2)$
3. The open interval $(2, 6)$.

We finish this section now with a digression about bounded and unbounded subsets of \mathbb{R} .

Basically, a subset X of \mathbb{R} is *bounded* if, on the number line, its elements are all enclosed within some region. In other words there exist real numbers a and b with $a < b$, for which all the points of X are in the interval (a, b) .

Definition 2.3.10. Let X be a subset of \mathbb{R} . Then X is bounded below if there exists a real number a with $a < x$ for all elements x of X . (Note that a need not belong to X here).

The set X is bounded above if there exists a real number b with $x < b$ for all elements x of X . (Note that b need not belong to X here).

The set X is bounded if it is bounded above and bounded below (otherwise it's unbounded).

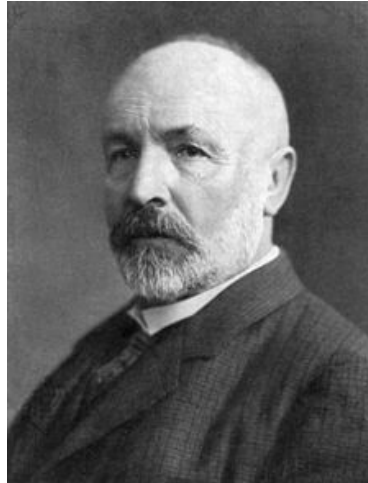
Example 2.3.11.

1. \mathbb{Q} is unbounded.
2. \mathbb{N} is bounded below but not above.
3. $(0, 1)$, $[0, 1]$, $[2, 100]$ and all open and closed intervals are bounded.
4. $\{\cos x : x \in \mathbb{R}\}$ is bounded, since $\cos x$ can only have values between -1 and 1 .

Remark: Example 2.3.8 shows that it is possible for a bounded subset of \mathbb{R} to have the same cardinality as the full set \mathbb{R} of real numbers.

2.4 \mathbb{R} is uncountable

Our goal in this section is to show that the set \mathbb{R} of real numbers is *uncountable* or *non-denumerable*; this means that its elements cannot be *listed*, or cannot be put in one-to-one correspondence with the natural numbers. We saw at the end of Section 1.3 that \mathbb{R} has the same cardinality as the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, or the interval $(-1, 1)$, or the interval $(0, 1)$. We will show that the open interval $(0, 1)$ is uncountable. This assertion and its proof date back to the 1890's and to Georg Cantor. The proof is often referred to as "Cantor's diagonal argument" and applies in more general contexts than we will see in these notes.



Georg Cantor : born in St Petersburg (1845), died in Halle (1918)

Theorem 2.4.1. *The open interval $(0, 1)$ is not a countable set.*

Before embarking on a proof, we recall precisely what this set is. It consists of all real numbers that are greater than zero and less than 1, or equivalently of all the points on the number line that are to the right of 0 and to the left of 1. It consists of all numbers whose decimal representation have only 0 before the decimal point (except 0.000... which is equal to 0, and 0.99999... which is equal to 1). Note that the digits after the decimal point may terminate in an infinite string of zeros, or may have a repeating pattern to their digits, or may not have either of these properties. The interval $(0, 1)$ includes all these possibilities.

Our goal is to show that the interval $(0, 1)$ cannot be put in bijective correspondence with the set \mathbb{N} of natural numbers. Our strategy is to show that no attempt at constructing a bijective correspondence between these two sets can ever be complete; it can never involve *all* the real numbers in the interval $(0, 1)$ no matter how it is devised. In order to achieve this, we will imagine that we had a listing of the elements of the interval $(0, 1)$; i.e. a bijective correspondence between this interval and \mathbb{N} . Such a correspondence would have to look something like the following.

\mathbb{N}	$(0, 1)$
1	\longleftrightarrow 0.13567324...
2	\longleftrightarrow 0.10000000...
3	\longleftrightarrow 0.32323232...
4	\longleftrightarrow 0.56834662...
5	\longleftrightarrow 0.79993444...
\vdots	\vdots

Note: The exact numbers that appear in the right-hand column above are not important, the point is that a bijective correspondence between \mathbb{N} and $(0, 1)$ would have this general form. We

don't know whether any particular decimal number in the right hand side terminates in zeros (or repeats) or not, but we know that some do and some don't.

So the entries in the right hand column above are basically infinite sequences of digits from 0 to 9. The right hand column must then consist somehow of a list of *all* such sequences. Our problem is to show that this is impossible : that no matter how the right hand column is constructed, it can't contain *every* sequence of digits from 1 to 9. We can do this by exhibiting an example of a sequence that can't possibly be there.

Suppose our list starts as follows.

N	↔	(0,1)
1	↔	0.13567324...
2	↔	0.10000000...
3	↔	0.32323232...
4	↔	0.56834662...
5	↔	0.79993444...
⋮		⋮

We will construct an element x of $(0,1)$ that is not in the list. To do so :

1. Look at the *first* digit after the decimal point in Item 1 in the list. If this is 1, write 2 as the first digit after the decimal point in x . Otherwise, write 1 as the first digit after the decimal point in x . So x differs in its first digit from Item 1 in the list.
2. Look at the *second* digit after the decimal point in Item 2 in the list. If this is 1, write 2 as the second digit after the decimal point in x . Otherwise, write 1 as the second digit after the decimal point in x . So x differs in its second digit from Item 2 in the list.
3. Look at the *third* digit after the decimal point in Item 3 in the list. If this is 1, write 2 as the third digit after the decimal point in x . Otherwise, write 1 as the third digit after the decimal point in x . So x differs in its third digit from Item 3 in the list.
4. Continue to construct x digit by digit in this manner. At the n th stage, look at the n th digit after the decimal point in Item n in the list. If this is 1, write 2 as the n th digit after the decimal point in x . Otherwise, write 1 as the n th digit after the decimal point in x . So x differs in its n th digit from Item n in the list.

What this process constructs is an element x of the interval $(0,1)$ that does not appear in the proposed list. The number x is not Item 1 in the list, because it differs from Item 1 in its 1st digit, it is not Item 2 in the list because it differs from Item 2 in its 2nd digit, it is not Item n in the list because it differs from Item n in its n th digit.

We conclude that the set of real numbers \mathbb{R} is *not countable* (or *uncountable*).

Note:

1. In our example, the number x would start 0.21111....
2. According to our construction, our x will always have all its digits equal to 1 or 2. So not only have we shown that the interval $(0,1)$ is uncountable, we have even shown that the set of all numbers in this interval whose digits are all either 1 or 2 is uncountable.
3. A challenging exercise : why would the same proof not succeed in showing that the set of rational numbers in the interval $(0,1)$ is uncountable?

Informally, Cantor's diagonal argument tells us that the "infinity" that is the cardinality of the real numbers is "bigger" than the "infinity" that is the cardinality of the natural numbers, or integers, or rational numbers. He was able to use the same argument to construct examples

of infinite sets of different (and bigger and bigger) cardinalities. So he actually established the notion of infinities of different magnitudes.

The work of Cantor was not an immediate hit within his own lifetime. It met some opposition from the finitist school which held that only mathematical objects that can be constructed in a finite number of steps from the natural numbers could be considered to exist. Foremost among the proponents of this viewpoint was Leopold Kronecker. From the book "The Honors Class" by Ben Yandell :

Many mathematicians, Leopold Kronecker in Berlin, in particular, were bothered by this headlong leap into the infinite, accessible only by inference, not finite construction. Georg Cantor (1845-1918), teaching at Halle in 1888, had invented set theory in the 1870s and was writing about infinities of different sizes and even doing arithmetic with them. But Kronecker would admit only numbers or other mathematical objects that were finitely 'constructible'.



Leopold Kronecker (1823-1891)

God made the integers, all else is the work of man.

What good your beautiful proof on π ? Why investigate such problems, given that irrational numbers do not even exist?

The work of Cantor had influential admirers too, among them David Hilbert, who set the course of much of 20th Century mathematics in his address to the International Congress of Mathematicians in Paris in 1900.



David Hilbert (1862-1943)

No one shall expel us from the paradise that Cantor has created for us.

What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

Hilbert's address to the Paris Congress is one of the most famous mathematical lectures ever. In it he posed 23 unsolved problems, the first of which was Cantor's *Continuum Hypothesis*. The Continuum Hypothesis proposes that every subset of \mathbb{R} is either countable (i.e. has the same cardinality as \mathbb{N} or \mathbb{Z} or \mathbb{Q}) or has the same cardinality as \mathbb{R} . This seems like a question to which the answer is either a straightforward yes or no, but it took the work of Kurt Gödel in the 1930s and Paul Cohen in the 1960s to reach the remarkable conclusion that the answer to the question is *undecidable*. This means essentially that the standard axioms of set theory do not provide enough structure to determine the answer to the question. Both the Continuum Hypothesis and its negation are consistent with the working rules of mathematics. People who work in set theory can legitimately assume that either the Continuum Hypothesis is satisfied or not. Fortunately most of us can get on with our mathematical work without having to worry about this very often.

2.5 The Completeness Axiom in \mathbb{R}

The rational numbers and real numbers are closely related, even though the set \mathbb{Q} of rational numbers is countable and the set \mathbb{R} of real numbers is not, and in this sense there are many more real numbers than rational numbers. However, \mathbb{Q} is “dense” in \mathbb{R} . This means that every interval of the real number line, no matter how short, contains infinitely many rational numbers. This statement has a practical impact as well, which we use all the time whether consciously or not.

Lemma 2.5.1. *Every real number (whether rational or not) can be approximated by a rational number with a level of accuracy as high as we like.*

Justification for this claim : 3 is a rational approximation for π .

3.1 is a closer one.

3.14 is closer again.

3.14159 is closer still.

3.1415926535 is even closer than that, and we can keep improving on this by truncating the decimal expansion of π at later and later stages. For example if we want a rational approximation that differs from the true value of π by less than 10^{-20} we can truncate the decimal approximation of π at the 21st digit after the decimal point. This is what is meant by “a level of accuracy as high as we like” in the statement of the lemma.

Notes:

1. The fact that all real numbers can be approximated with arbitrary closeness by rational numbers is used all the time in everyday life. Computers basically don't deal with all the real numbers or even with all the rational numbers, but with some specified level of precision. They really work with a subset of the rational numbers.

2. The *sequence*

3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, ...

is a list of numbers that are steadily approaching π . All of these numbers are less than π ; they are increasing and they are approaching π . We say that this sequence *converges* to π and we will investigate the concept of convergent sequences in Chapter 2.

3. We haven't looked yet at the question of how the numbers in the above sequence can be calculated, i.e. how we can get our hands on better and better approximations to the value of the irrational number π . That's another thing that we will look at in Chapter 2.

The goal of this last section of Chapter 1 is to pinpoint one essential property of subsets of \mathbb{R} that is not shared by subsets of \mathbb{Z} or of \mathbb{Q} . We need a few definitions and some terminology in order to describe this.

Definition 2.5.2. *Let S be a subset of \mathbb{R} . An element b of \mathbb{R} is an upper bound for S if $x \leq b$ for all $x \in S$. An element a of \mathbb{R} is a lower bound for S if $a \leq x$ for all $x \in S$.*

So an upper bound for S is a number that is to the right of all elements of S on the real line, and a lower bound for S is a number that is to the left of all points of S on the real line. Note that if b is an upper bound for S , then so is every number b' with $b < b'$. If a is a lower bound for S then so is every number a' with $a' < a$. So if S has an upper bound at all it has infinitely many upper bounds, and if S has a lower bound at all it has infinitely many lower bounds. Recall that

- S is *bounded above* if it has an upper bound,
- S is *bounded below* if it has a lower bound,
- S is *bounded* if it is bounded both above and below.

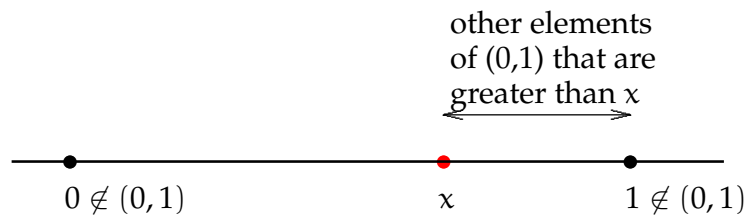
In this section we are mostly interested in sets that are bounded on at least one side.

Definition 2.5.3. Let S be a subset of \mathbb{R} . If there is a number m that is both an element of S and an upper bound for S , then m is called the **maximum element** of S and denoted $\max(S)$.

If there is a number ι that is both an element of S and a lower bound for S , then ι is called the **minimum element** of S and denoted by $\min(S)$.

Notes

1. A set can have at most one maximum (or minimum) element. For suppose that both m and m' are maximum elements of S according to the definition. Then $m' \leq m$ because m is a maximum element of S , and $m \leq m'$ because m' is a maximum element of S . The only way that both of these statements can be true is if $m = m'$.
2. Pictorially, on the number line, the maximum element of S is the rightmost point that belongs to S , if such a point exists. The minimum element of S is the leftmost point on the number line that belongs to S , if such a point exists.
3. There are basically two reasons why a subset S of \mathbb{R} might fail to have a maximum element. First, S might not be bounded above - then it certainly won't have a maximum element. Secondly S might be bounded above, but might not contain an element that is an upper bound for itself. Probably the easiest example of this to think about is an open interval like $(0,1)$. This set is certainly bounded above - for example by 1. However, take any element x of $(0,1)$. Then x is a real number that is strictly greater than 0 and strictly less than 1. Between x and 1 there are more real numbers all of which belong to $(0,1)$ and are greater than x . So x is not an upper bound for the interval $(0,1)$.



An open interval like $(0,1)$, although it is bounded, has no maximum element and no minimum element.

An example of a subset of \mathbb{R} that *does* have a maximum and a minimum element is a *closed* interval like $[2,3]$. The minimum element of $[2,3]$ is 2 and the maximum element is 3.

Remark : Every *finite* subset of \mathbb{R} has a maximum element and a minimum element (these may be the same if the set has only one element).

For bounded subsets of \mathbb{R} , there are notions called the *supremum* and *infimum* that are closely related to maximum and minimum. Every subset of \mathbb{R} that is bounded above has a supremum and every subset of \mathbb{R} that is bounded below has an infimum. This is the *Axiom of Completeness* for \mathbb{R} .

Definition 2.5.4. Let S be a subset of \mathbb{R} that is bounded above. Then the set of all upper bounds for S has a minimum element. This number is called the **supremum** of S and denoted $\sup(S)$.

Let S be a subset of \mathbb{R} that is bounded below. Then the set of all lower bounds for S has a maximum element. This number is called the **infimum** of S and denoted $\inf(S)$.

Notes

1. The *supremum* of S is also called the *least upper bound (lub)* of S . It is the least of all the numbers that are upper bounds for S .
2. The *infimum* of S is also called the *greatest lower bound (glb)* of S . It is the greatest of all the numbers that are lower bounds for S .

3. Definition 2.5.4 is simultaneously a definition of the terms *supremum* and *infimum* and a statement of the *Axiom of Completeness* for the real numbers.

To see why this statement says something special about the real numbers, temporarily imagine that the only number system available to us is \mathbb{Q} , the set of rational numbers. Look at the set

$$S := \{x \in \mathbb{Q} : x^2 < 2\}.$$

So S consists of all those rational numbers whose square is less than 2. It is bounded below, for example by -2 , and it is bounded above, for example by 2. This is saying that every rational number whose square is less than 2 is itself between -2 and 2 (of course we could narrow this interval with a bit more care). The positive elements of S are all those positive rational numbers that are less than the real number $\sqrt{2}$.

Claim 1: S does not have a maximum element.

To see this, suppose that x is a candidate for being the maximum element of S . Then x is a rational number and $x^2 \leq 2$. For any (very large) integer n , $x + \frac{1}{n}$ is a rational number and

$$\left(x + \frac{1}{n}\right)^2 = x^2 + 2\frac{x}{n} + \frac{1}{n^2}.$$

We can choose n large enough that $2\frac{x}{n} + \frac{1}{n^2}$ is so small that $\left(x + \frac{1}{n}\right)^2$ is still less than 2. Then the number $x + \frac{1}{n}$ belongs to S , and it is bigger than x , contrary to the hypothesis that x could be a maximum element of S .

Claim 2: S has no least upper bound in \mathbb{Q} .

To see this, suppose that x is a candidate for being a least upper bound for S in \mathbb{Q} . Then $x^2 > 2$. Note x^2 cannot be equal to 2 because $x \in \mathbb{Q}$.

For a (large) integer n

$$\left(x - \frac{1}{n}\right)^2 = x^2 - 2\frac{x}{n} + \frac{1}{n^2} = x^2 - \frac{1}{n} \left(2x - \frac{1}{n}\right).$$

Choose n large enough that $x^2 - \frac{1}{n} \left(2x - \frac{1}{n}\right)$ is still greater than 2. Then $x - \frac{1}{n}$ is still an upper bound for S , and it is less than x .

So S has no least upper bound in \mathbb{Q} .

If we consider the same set S as a subset of \mathbb{R} , we can see that $\sqrt{2}$ is the supremum of S in \mathbb{R} (and $-\sqrt{2}$ is the infimum of S in \mathbb{R}).

This example demonstrates that the *Axiom of Completeness* does not hold for \mathbb{Q} , i.e. a bounded subset of \mathbb{Q} need not have a supremum in \mathbb{Q} or an infimum in \mathbb{Q} .

Example 2.5.5 (Summer Examinations 2011). *Determine with proof the supremum and infimum of the set*

$$T = \left\{5 - \frac{5}{n} : n \in \mathbb{N}\right\}.$$

Solution: (Supremum) First, look at the numbers in the set. All of them are less than 5. They can be very close to 5 if n is large.

Guess: $\sup(T) = 5$.

We need to show :

1. 5 is an upper bound for T .

To see this, suppose that $x \in T$. Then $x = 5 - \frac{5}{k}$ for some $k \in \mathbb{N}$. Since k is positive, $\frac{5}{k}$ is positive and $x < 5$. Hence 5 is an upper bound for T .

2. If b is an upper bound for T , then b cannot be less than 5.
To see this suppose that $b < 5$, so $5 - b$ is a positive real number. We can choose a natural number m so large that $\frac{5}{m} < 5 - b$. Then $5 - \frac{5}{m} > b$, which means that b is not an upper bound for T , as $5 - \frac{5}{m} \in T$.

Thus 5 is the *least upper bound (supremum)* of T .

Solution: (Infimum) Look for the least elements of T . These occur when n is small : when $n = 1$ we get that $5 - \frac{5}{1} = 0$ belongs to T .

Guess: $\inf(T) = 0$.

We need to show :

1. 0 is a lower bound for T .
To see this, suppose that $x \in T$. Then $x = 5 - \frac{5}{k}$ for some $k \in \mathbb{N}$. Since $k \in \mathbb{N}$, $k \geq 1$ and $\frac{5}{k} \leq 5$. Thus $5 - \frac{5}{k} \geq 5 - 5$ which means $x \geq 0$ and 0 is a lower bound for T .
2. If a is a lower bound for T , then a cannot be greater than 0.
No number greater than 0 can be a lower bound for T , since $0 \in T$. Thus 0 is the *minimum element* (and therefore the infimum) of T .

2.6 Exam advice and sample questions for Chapter 2

This section is intended to provide some learning resources for the content of Chapter 2, with a view towards the Summer Exam. The second calculus question on the exam will be based on Chapter 2. The purpose of this question (from the point of view of the examiners) is to assess how well students have achieved the learning outcomes for Chapter 2. These learning outcomes are listed at the end of each section of the Lecture notes, and in Chapter 2 they are generally concerned with properties of sets of real numbers such as cardinality, boundedness and countability. There are a few sample exam questions given in this section. The first one is accompanied by a solution that would be considered appropriate and fully correct in an exam context, in terms of the level of explanation and detail provided. In the exam it is important to give a clear and readable account of what you are doing as well as to clearly state your answer.

Don't read too much into these sample questions - they are intended to give you a sense of the style of a typical exam question, but that's all.

SAMPLE "EXAM-TYPE" QUESTIONS

1. (from the 2013 Summer Exam)

- (a) For each of the following subsets of \mathbb{R} , determine whether it is bounded, countable or neither:
(i) \mathbb{R} (ii) \mathbb{Q} (iii) $\{x \in \mathbb{Q} : x^2 \leq 2\}$ (iv) $\{x \in \mathbb{R} : x^2 < 1\}$.
- (b) By finding a suitable bijection, or otherwise, show that $|(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$. Deduce that $|(a, b)| = |\mathbb{R}|$ for all $a < b \in \mathbb{R}$.
- (c) Determine, with proof, the supremum and infimum of the set

$$\left\{ \frac{n+1}{n} : n \in \mathbb{N} \right\}.$$

Solution

- (a) (i) Neither bounded nor countable.
(ii) Countable and not bounded.
(iii) Bounded and countable.
(iv) Bounded and uncountable.
- (b) The function \tan provides a bijective correspondence between the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and \mathbb{R} , establishing that these sets have the same cardinality.

Suppose that $a < b$ in \mathbb{R} . Then $b - a > 0$ and we may define a bijective correspondence between (a, b) and $(0, b - a)$ by $x \leftrightarrow x - a$. We may define a bijective correspondence between $(0, b - a)$ and $(0, \pi)$ by $x \leftrightarrow \frac{\pi}{b-a}x$. Finally we may define a bijective correspondence between $(0, \pi)$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ by $x \leftrightarrow x - \frac{\pi}{2}$. This shows that for all $a < b$ the open interval (a, b) has the same cardinality as $(-\frac{\pi}{2}, \frac{\pi}{2})$ and hence as \mathbb{R} .

- (c) Write $S = \{\frac{n+1}{n} : n \in \mathbb{N}\}$.

$\sup(S) = 2$. To see this note first that $1 \in S$ since $2 = \frac{1+1}{1}$. Let $x \in S$. Then $x = \frac{m+1}{m} = 1 + \frac{1}{m}$ for some positive integer m . Since $\frac{1}{m} \leq 1$, $x \leq 1 + 1 \implies x \leq 2$. Then 2 is both an element of S and an upper bound for S which means $\sup(S) = \max(S) = 2$.

$\inf(S) = 1$. To see this note first that 1 is a lower bound for S , since every element of S has the form $1 + \frac{1}{n}$ for a natural number n and is therefore greater than 1. Suppose that $a > 1$. Then (no matter how close to 1 a is) it is possible to choose a natural number m for which $\frac{1}{m} < a - 1$. Then $1 + \frac{1}{m} < a$, which means that a is not a lower bound for S . Thus 1 is the greatest lower bound of S and $\inf(S) = 1$.

2. (a) Give an example of
- a subset of \mathbb{R} that is bounded and uncountable;
 - a subset of \mathbb{R} that is countable and bounded above but not below;
 - a subset of \mathbb{R} that is bounded and has no maximum element.
- (b) State what it means for an infinite set to be *countable*. Outline Cantor's "diagonal argument" to show that the interval $(0, 1)$ is uncountable.
- (c) Determine, with explanation, the supremum and infimum of the set

$$\left\{ \frac{2}{n^2 + 1}, n \in \mathbb{Z} \right\}.$$

3. (a) Determine whether each of the following statements is true or false. Give an explanation for your answer.
- The set of irrational real numbers is a countable set.
 - Every subset of \mathbb{Q} is countable.
 - Every countable subset of \mathbb{R} is a subset of \mathbb{Q} .
- (b) Explain what it means for an infinite set to be *countable*. Show that the set of even integers is a countable set.
- (c) Show that the set S defined below is bounded and determine its infimum and supremum, with explanation.

$$S = \left\{ \frac{2n + 1}{n^3}, n \in \mathbb{N} \right\}.$$