Cardinality

Definition

The cardinality of a finite set S, denoted $|S|$, is the number of elements in S.

Example

1. If $S = \{5, 7, 8\}$ then $|S| = 3$.
2. $|\{4, 10, \pi\}| = 3$
3. $|\{x \in \mathbb{Z} : \pi < x < 3\pi\}| = 6$.
   Note: $\{x \in \mathbb{Z} : \pi < x < 3\pi\} = \{4, 5, 6, 7, 8, 9\}$.
4. The cardinality of $\mathbb{Q}$ is infinite.

Remarks

1. The notation "$|\cdot|$" is severely overused in mathematics. If x is a real number, $|x|$ means the absolute value of x. If S is a set, $|S|$ means the cardinality of S. If A is a matrix $|A|$ means the determinant of A. It is supposed to be clear from the context what is meant.
2. Defining the concept of cardinality for infinite sets is trickier, since you can't say how many elements they have. We will be able to say though what it means for two infinite sets to have the same (or different) cardinalities.

A silly example

Example

In a hotel, keys for all the guest rooms are kept on hooks behind the reception desk. If a room is occupied, the key is missing from its hook because the guests have it. If the receptionist wants to know how many rooms are occupied, s/he doesn't have to visit all the rooms to check - s/he can just count the number of hooks whose keys are missing.

In this example, the occupied rooms are in one-to-one correspondence with the empty hooks. This means that each occupied room corresponds to one and only one empty hook, and each empty hook corresponds to one and only one occupied room. So the number of empty hooks is the same as the number of occupied rooms and we can count one by counting the other.

Bijections and bijective correspondence

Definition

Suppose that A and B are sets. Then a one-to-one correspondence or a bijective correspondence between A and B is a pairing of each element of A with an element of B, in such a way that every element of B is matched to exactly one element of A.

Definition

Suppose that A and B are sets. A function $f : A \longrightarrow B$ is called a bijection if

- *Whenever a_1 and a_2 are different elements of A, $f(a_1)$ and $f(a_2)$ are different elements of B.*
- *Every element b of B is the image of some element a of A.*

Cardinality and bijective correspondence

If a bijective correspondence exists between two finite sets, they have the same cardinality. Sometimes, in order to determine the cardinality of a set, it is easiest to determine the cardinality of another set with which we know it is in bijective correpsondence.

Example

How many integers between 1 and 1000 are perfect squares?

Solution: The list of perfect squares in our range begins as follows

$$1,\ 4,\ 9,\ 16, \dots$$

One way to proceed would be to keep writing out successive terms of this sequence until we hit one that exceeds 1000, then delete that one and count the terms that we have. This is more work than we are asked to do, since we don't need the list of squares but just the number of them. Alternatively, we could notice that $(31)^2 = 961$ and $(32)^2 = 1024$. So the numbers $1^2, 2^2, \dots, (31)^2$ are the only squares in the range 1 to 1000 and there are 31 of them.

Another example of bijective correspondence

This last example shows that it could be possible to know that there is a bijective correspondence between two finite sets, without knowing the cardinality of either of them.

Example

Show that the equations

$$x^3 + 2x + 4 = 0 \textit{ and } x^3 + 3x^2 + 5x + 7 = 0$$

have the same number of real solutions.

$x^3 + 2x + 4 = 0$ and $x^3 + 3x^2 + 5x + 7 = 0$

Solution: One way of doing this is to demonstrate a bijective correspondence between their sets of real solutions. We can write

$$\begin{aligned} x^3 + 3x^2 + 5x + 7 &= (x^3 + 3x^2 + 3x + 1) + 2x + 6 \\ &= (x+1)^3 + (2x+2) + 4 \\ &= (x+1)^3 + 2(x+1) + 4. \end{aligned}$$

This means that a real number a is a solution of the second equation if and only if

$$(a+1)^3 + 2(a+1) + 4 = 0$$

i.e. if and only if $a + 1$ is a solution of the first equation.
The correspondence $a \longleftrightarrow a+1$ is a bijective correspondence between the solution sets of the two equations. So they have the same number of real solutions.
Note: This number is at least 1 and at most 3. Why?

Learning outcomes for Section 2.2

After studying this section you should be able to

- Explain what is meant by the cardinality of a set;
- Read and interpret descriptions of different subsets of $\mathbb{R}$ presented using different standard notations. Decide what the elements of these sets are and whether the sets are finite or infinite;
- Explain what is meant by a *bijective correspondence* and give examples to support your explanation.