(a) There may be a unique solution. This will happen if all variables are leading variables, i.e. every column except the rightmost one in a REF obtained from the augmented matrix has a leading 1. In the case the *reduced* row-echelon form obtained from the augmented matrix will have the following form :

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

with possibly some additional rows full of zeroes at the bottom. The unique solution can be read from the right-hand column.

NOTE: If a system of equations has a unique solution, the number of equations must be at least equal to the number of variables (since the augmented matrix must have enough rows to accommodate a leading 1 for every variable).

(b) There may be infinitely many solutions. This happens if the system is consistent but at least one of the variables is free. In this case the number of leading 1s in the row echelon form is less than the number of variables in the system.

1.4 Gaussian Elimination and Matrix Algebra

We finish Chapter 1 with two observations about connections between the process of Gaussian (or Gauss-Jordan) elimination and the algebra of matrices as objects that can be added, multipied, inverted etc.

The first is that elementary row operations may themselves be interpreted as matrix multiplication exercises. We write I_m for the $m \times m$ identity matrix and $E_{i,j}$ for the matrix that has 1 in the (i, j)-position and eros everywhere else. So for example $I_3 + 4E_{1,2}$ is the 3×3 matrix with entries 1 on the main diagonal, 4 in the $(1, 2)$ position, and zeros everywhere else.

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 1.4.1. *Let A be a $m \times m$ matrix. Then elementary row operations on A amount to multiplying A on the left by $m \times m$ matrices, as follows:*

1. *Mutiplying Row i by the non-zero scalar α is equivalent to multiplying A on the left by the matrix $I_m + (\alpha - 1)E_{i,i}$, which has entries α in Position (i, i), 1 in all other positions on the main diagonal, and zeros in all off-diagonal positions.*

2. *Switching Rows i and k amounts to multiplying A on the left by the matrix $I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,i}$. This matrix has entries 1 in the (i, k) and (k, i) positions, and in the (j, j) position for all $j \notin \{i, k\}$, and zeros elsewhere. It has exactly one 1 in each row and column, and is otherwise full of zeros.*

3. *Adding $\alpha \times$ Row i to Row k amounts to multiplying A on the left by the matrix $I_m + \alpha E_{k,i}$, which has α in the (k, i) position, entries 1 on the main diagonal, and zeros elswhere.*

Here are a couple of examples.

1. $$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 3 & -5 & -7 \\ 2 & 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 2 & 2 & 3 & 2 \\ 2 & 3 & -1 & 1 \\ 1 & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix}$$

Matrices of the three types described in Theorem 1.4.1 are sometimes referred to as *elementary matrices*. They are always invertible, and their inverses are also elementary matrices. The statement that every matrix can be reduced to RREF through a sequence of EROs is equivalent to saying that for every matrix A with m rows, there exists a $m \times m$ matrix B, which is a product of elementary matrices, with the property that BA is in RREF.

Exercise 1.4.2. *Write down the inverse of an elementary matrix of each of the three types, and show that it is also an elementary matrix.*
(Hint: Think about how to reverse an elementary row operation, with another elementary row operation).

Exercise 1.4.3. *Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.*

The second point of this section is that not only can elementary row operations be interpreted as matrix products, but they can also be used for calculations in matrix algebra beyond the context of solving systems of linear equations. As an example of this, we note that Gauss-Jordan elimination can be used to calculate the inverse of a square matrices (and this is a much more efficient method than calculating cofactors as we often do for 3×3 matrices). Suppose that $A \in M_n(\mathbb{F})$, for some field $\mathbb{F}$. If A is invertible, let $v_1, v_2, \dots, v_n$ be the columns of its inverse. Then

$$AA^{-1} = A \begin{bmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{bmatrix} = A \begin{bmatrix} | & | & \dots & | \\ Av_1 & Av_2 & \dots & Av_n \\ | & | & \dots & | \end{bmatrix} I_n.$$

It follows that for each i, Av_i is the ith column of the identity matrix, which has 1 in position i and zeros elsewhere. This means that v_i is the solution of the linear system $Av_i = e_i$, where e_i is column i of the identity matrix, and the variables are the unknown entries of v_i. Since A is invertible, this system has the unique solution $v_i = A^{-1}e_i$, and this unique solution cn be found by applying Gauss-Jordan elimination to the augmented matrix of the system, which is $[A|e_i]$. We need to this for each column, but we can combine this into a single process by writing $e_1, e_2, \dots, e_n$ as n distinct columns in the "right hand side" of a $n \times 2n$ augmented matrix whose coefficient matrix is A. If A is invertible, the RREF obtained from this augmented matrix has leading 1s in the first n columns, which form a copy of I_n, and the inverse of A is written in the last n columns of the RREF.

Example 1.4.4. *Find A^{-1} if* $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

To calculate A^{-1}, We start with the 3×6 matrix

$$A' = \begin{bmatrix} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{bmatrix}.$$

Reduce A' to RREF. If the RREF has I_3 in its first three columns, then columns 4,5,6 contain A^{-1}. If the RREF does not have leading 1s in its first three columns, we conclude that A is not invertible (more later on the justification for this). We proceed as follows.

$$\left[\begin{array}{cccccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \quad \overset{R1 \leftrightarrow R2}{\longrightarrow} \quad \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R3 \to R3 - 2R1}{\overset{R2 \to R2 - 3R1}{\longrightarrow}} \quad \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] \quad \overset{R3 \to R3 - R2}{\longrightarrow} \quad \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\overset{R3 \leftrightarrow R2}{\longrightarrow} \quad \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{array}\right] \quad \overset{R3 \leftrightarrow R3 - 4R2}{\longrightarrow} \quad \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$$

$$\overset{R3 \times (-\frac{1}{10})}{\longrightarrow} \quad \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right] \quad \overset{R1 \to R1 - 3R3}{\longrightarrow} \quad \left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

The above matrix is in RREF and its first three columns comprise I_3. We conclude that the matrix A^{-1} is written in the last three columns, i.e.

$$A^{-1} = \left[\begin{array}{ccc} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right].$$

It is easily checked that $AA^{-1} = I_3$.