Example 1.2.15. *If* $A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & 4 \end{pmatrix}$, *then* $A^T = \begin{pmatrix} 1 & 2 \\ -2 & 0 \\ -3 & 4 \end{pmatrix}$.

For all relevant i and j, the (i,j) entry of A^T is the (j,i) entry of A. If A is $m \times n$, then the products AA^T and A^TA always exist, and they are square matrices of size $m \times m$ and $n \times n$ respectively. Moreover, they are *symmetric*. A square matrix is symmetric if it is equal to its own transpose.

In the above example,

$$AA^T = \begin{pmatrix} 14 & -10 \\ -10 & 20 \end{pmatrix}, \quad A^TA = \begin{pmatrix} 5 & -2 & 5 \\ -2 & 4 & 6 \\ 5 & 6 & 25 \end{pmatrix}.$$

The following lemma describes how the transpose of a matrix product AB depends on the transposes of A and B. The proof is included as an example of how demonstrations in matrix theory often involve the close inspection and description of entries in particular positions. They are fussy and not easy to read, but they often don't involve anything more than careful attention to the details (all the information is contained in the subscripts!). It takes a bit of concentration to get used to arguments like this and to write them (and read them), but you are encouraged to study this one and satisfy yourself that you could produce it yourself if necessary.

Lemma 1.2.16. *Let* A *and* B *be matrices for which the product* AB *is defined. Then* $(AB)^T = B^TA^T$.

The lemma is saying that the transpose of the product A is the product of the transposes of A and B, but in the opposite order. To prove this, we analyse the entry in an arbitrary position (i,j) of $(AB)^T$, noting that this is the (j,i)-entry of AB.

Proof. Suppose that the sizes of A and B are $m \times p$ and $p \times n$ respectively. Choose an arbitrary position (i,j) in $(AB)^T$. The entry in this position is

$$\begin{aligned} (AB)^T_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{p} A_{jk}B_{ki} \\ &= \sum_{k=1}^{p} B^T_{ik}A^T_{kj} \\ &= (B^TA^T)_{ij}. \end{aligned}$$

□

If you are thinking about this proof, note that the third last line describes how the (j,i)-entry of AB is assembled from Row j of A and Column i of B. The next line rewrites that in terms of entries of B^T and A^T. Note that $B^T_{ik} = B_{ki}$ and $A^T_{kj} = A_{jk}$.

Exercise 1.2.17. *Use Lemma 1.2.16 to explain why the matrices* AA^T *and* A^TA *are always symmetric.*

1.3 Systems of Linear Equations

Consider the equation

$$2x + y = 3.$$

This is an example of a *linear equation* in the variables x and y. As it stands, the statement "$2x + y = 3$" is neither true nor untrue : it is just a statement involving the abstract symbols x and y. However if we replace x and y with some particular pair of real numbers, the statement will become either true or false. For example

Putting $x = 1,\ y = 1$ gives $2x + y = 2(1) + (1) = 3$: True
$x = 1,\ y = 2$ gives $2x + y = 2(1) + (2) \neq 3$: False
$x = 0,\ y = 3$ gives $2x + y = 2(0) + (3) = 3$: True

Definition 1.3.1. *A pair (x_0, y_0) of real numbers is a* solution *to the equation $2x+y = 3$ if setting $x = x_0$ and $y = y_0$ makes the equation true; i,.e. if $2x_0 + y_0 = 3$.*

For example $(1, 1)$ and $(0, 3)$ are solutions - so are $(2, -1)$, $(3, -3)$, $(-1, 5)$ and $(-1/2, 4)$ (check these).

However $(1, 4)$ is not a solution since setting $x = 1,\ y = 4$ gives $2x + y = 2(1) + 4 \neq 3$.

The set of all solutions to the equation is called its *solution set*. In tis example, the solution set is a *line* in $\mathbb{R}^2$. In general, the solution set of the linear equation

$$a_1X_1 + \cdots + a_nX_n = b,$$

where b and the a_i are real numbers (and the a_i are not all zero) is an *affine hyperplane* in $\mathbb{R}^n$; geometrically it resembles a copy of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$.

A collection of linear equations in the same n variables is referred to as a *linear system* or *system of linear equations*. The solution set of the system is the subset of $\mathbb{R}^n$ consisting of those elements that satify all of the equations in the system; it is the intersection of the solution sets of the individual equations. For small systems in few variables, like the one below, the solution set can be easily computed.

Example 1.3.2. *Solve the linear system*

$$\left.\begin{array}{rcrcrl} 2x & + & y & = & 3 & (A) \\ 4x & + & 3y & = & 4 & (B) \end{array}\right\}$$

Step 1: Multiply Equation (A) by $2 : 4x + 2y = 6$ (A2).
Any solution of (A2) is a solution of (A).

Step 2: Multiply Equation (B) by $-1 : -4x - 3y = -4$ (B2)
Any solution of (B2) is a solution of (B).

Step 3: Now add equations (A2) and (B2).

$$\begin{array}{rcrcr} 4x & + & 2y & = & 6 \\ -4x & - & 3y & = & -4 \\ \hline \\ & & -y & = & 2 \end{array}$$

Step 4: So $y = -2$ and the value of y in any simultaneous solution of (A) and (B) is -2 : Now we can use (A) to find the value of x.

$$\begin{aligned} 2x + y = 3 \text{ and } y = -2 \Longrightarrow 2x + (-2) &= 3 \\ \Longrightarrow 2x &= 5 \\ \Longrightarrow x &= \frac{5}{2} \end{aligned}$$

So $x = 5/2,\ y = -2$ is the *unique* solution to this system of linear equations.

No surprises there, but this kind of "ad hoc" approach may not be so easy if we have a more complicated system, involving a greater number of variables, or more equations. We will devise a systematic approach, known as Gauss-Jordan elimination, for solving systems of linear equations.

1.3.1 Elementary Row Operations

Example 1.3.3. *Find all solutions of the following system :*

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$$

In other (perhaps simpler) examples we were able to find solutions by simplifying the system (perhaps by eliminating certain variables) through operations of the following types :

1. We could multiply one equation by a non-zero constant.
2. We could add one equation to another (for example in the hope of eliminating a variable from the result).

A similar approach will work for Example 1.3.3 but with this and other harder examples it may not always be clear how to proceed. We now develop a new technique both for describing our system and for applying operations of the above types more systematically and with greater clarity.

Back to Example 1.3.3: We associate a *matrix* to our system of equations.

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix} \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array}$$

Note that the first *row* of this matrix contains as its four entries the coefficients of the variables x, y, z, and the number appearing on the right-hand-side of Equation 1 of the system. Rows 2 and 3 correspond similarly to Equations 2 and 3. The *columns* of the matrix correspond (from left to right) to the variables x, y, z and the right hand side of our system of equations.

Definition 1.3.4. *The above matrix is called the* augmented matrix *of the system of equations in Example 1.3.3.*

In solving systems of equations we are allowed to perform operations of the following types:

1. Multiply an equation by a non-zero constant.
2. Add one equation (or a non-zero constant multiple of one equation) to another equation.

These correspond to the following operations on the augmented matrix :

1. Multiply a *row* by a non-zero constant.
2. Add a multiple of one row to another row.
3. We also allow operations of the following type : Interchange two rows in the matrix (this only amounts to writing down the equations of the system in a different order).

Definition 1.3.5. *Operations of these three types are called* Elementary Row Operations (ERO's) *on a matrix.*

We now describe how ERO's on the augmented matrix can be used to solve the system of Example 1.3.3. The following table describes how an ERO is performed at each step to produce a new augmented matrix corresponding to a new (hopefully simpler) system.

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$
1. $R3 \to R3 + R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ & & 6y & + & z & = & 5 \end{array}$
2. $R2 \to R2 - 3R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & - & 5y & + & z & = & -6 \\ & & 6y & + & z & = & 5 \end{array}$
3. $R2 \to R2 + R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$
4. $R3 \to R3 - 6R2$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -11 & 11 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & -11z & = & 11 \end{array}$
5. $R3 \times \left(-\frac{1}{11}\right)$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcrl} x & + & 2y & - & z & = & 5 & (A) \\ & & y & + & 2z & = & -1 & (B) \\ & & & & z & = & -1 & (C) \end{array}$

We have produced a new system of equations. This is easily solved :

Backsubstitution
$$\begin{cases} (C) & z = -1 \\ (B) & y = -1 - 2z \quad \Longrightarrow \quad y = -1 - 2(-1) = 1 \\ (A) & x = 5 - 2y + z \quad \Longrightarrow \quad x = 5 - 2(1) + (-1) = 2 \end{cases}$$

Solution : $x = 2,\ y = 1, z = -1$

You should check that this is a solution of the original system. It is the only solution both of the final system and of the original one (and every intermediate one).

NOTE : The matrix obtained in Step 5 above is in *Row-Echelon Form*. This means :

1. The first non-zero entry in each row is a 1 (called a *Leading 1*).
2. If a column contains a leading 1, then every entry of the column below the leading 1 is a zero.
3. As we move downwards through the rows of the matrix, the leading 1's move from left to right.
4. Any rows consisting entirely of zeroes are grouped together at the bottom of the matrix.

NOTE : The process by which the augmented matrix of a system of equations is reduced to row-echelon form is called *Gaussian Elimination*. In Example 1.3.3 the solution of the system was found by Gaussian elimination with *Backsubstitution*.

General Strategy to Obtain a Row-Echelon Form

1. Get a 1 as the top left entry of the matrix.
2. Use this first leading 1 to "clear out" the rest of the first column, by adding suitable multiples of Row 1 to subsequent rows.
3. If column 2 contains non-zero entries (other than in the first row), use ERO's to get a 1 as the second entry of Row 2. After this step the matrix will look like the following (where the entries represented by stars may be anything):

$$\begin{pmatrix} 1 & * & * & \dots & \dots \\ 0 & 1 & \dots & \dots & \dots \\ 0 & * & \dots & \dots & \dots \\ 0 & * & \dots & \dots & \dots \\ \vdots & \vdots & & & \vdots \\ 0 & * & \dots & \dots & \dots \end{pmatrix}$$

4. Now use this second leading 1 to "clear out" the rest of column 2 (below Row 2) by adding suitable multiples of Row 2 to subsequent rows. After this step the matrix will look like the following :

$$\begin{pmatrix} 1 & * & * & \dots & \dots \\ 0 & 1 & * & \dots & \dots \\ 0 & 0 & * & \dots & \dots \\ 0 & 0 & * & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \dots & \dots \end{pmatrix}$$

5. Now go to column 3. If it has non-zero entries (other than in the first two rows) get a 1 as the third entry of Row 3. Use this third leading 1 to clear out the rest of Column 3, then proceed to column 4. Continue until a row-echelon form is obtained.

Example 1.3.6. *Let* A *be the matrix*

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$$

Reduce A *to row-echelon form.*

Solution:

1. Get a 1 as the first entry of Row 1. Done.

2. Use this first leading 1 to clear out column 1 as follows :

$$\begin{array}{ccc} & & \\ R2 & \to & R2 - 2R1 \\ R3 & \to & R3 - R1 \end{array} \quad \left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 0 & 3 & 1 & -2 & 8 \\ 0 & -2 & 3 & 5 & 2 \end{array}\right)$$

3. Get a leading 1 as the second entry of Row 2, for example as follows :

$$R2 \to R2 + R3 \quad \left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & -2 & 3 & 5 & 2 \end{array}\right)$$

4. Use this leading 1 to clear out whatever appears below it in Column 2 :

$$R3 \to R3 + 2R2 \quad \left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 11 & 11 & 22 \end{array}\right)$$

5. Get a leading 1 in Row 3 :

$$R3 \times \frac{1}{11} \quad \left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

This matrix is now in row-echelon form.
Remark Starting with a particular matrix, different sequences of ERO's can lead to different row-echelon forms. However, all have the same number of non-zero rows.

1.3.2 The Reduced Row-Echelon Form (RREF)

Definition 1.3.7. *A matrix is in* reduced row-echelon form (RREF) *if*

1. *It is in row-echelon form, and*
2. *If a particular column contains a leading 1, then* all *other entries of that column are zeroes.*

If we have a row-echelon form, we can use ERO's to obtain a reduced row-echelon form (using ERO's to obtain a RREF is called *Gauss-Jordan elimination*).

Example 1.3.8. *In Example 1.3.6, we obtained the following row-echelon form :*

$$\left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \quad \text{(REF, not } \textit{reduced} \text{ REF)}$$

To get a RREF from this REF :

1. Look for the leading 1 in Row 2 - it is in column 2. Eliminate the non-zero entry *above* this leading 1 by adding a suitable multiple of Row 2 to Row 1.

$$R1 \to R1 + R2 \quad \left(\begin{array}{rrrrr} 1 & 0 & 3 & 5 & 10 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

2. Look for the leading 1 in Row 3 - it is in column 3. Eliminate the non-zero entries *above* this leading 1 by adding suitable multiples of Row 3 to Rows 1 and 2.

$$\begin{array}{ccc} & & \\ R1 & \to & R1 - 3R3 \\ R2 & \to & R2 - 4R3 \end{array} \quad \left(\begin{array}{rrrrr} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

This matrix is in *reduced* row-echelon form. The technique outlined in this example will work in general to obtain a RREF from a REF: you should practise with similar examples.
Remark: Different sequences of ERO's on a matrix can lead to different row-echelon forms. However, the *reduced* row-echelon form of any matrix is unique.

1.3.3 Leading Variables and Free Variables

Example 1.3.9. *Find the general solution of the following system :*

$$\begin{array}{rcrcrcrcrl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \quad \text{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \quad \text{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \quad \text{III} \end{array}$$

SOLUTION :

1. Write down the augmented matrix of the system :

$$\begin{array}{l} \text{Eqn I} \\ \text{Eqn II} \\ \text{Eqn III} \end{array} \quad \left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array}\right)$$
$$\quad x_1 \quad x_2 \quad x_3 \quad x_4$$

   Note : This is the matrix of Example 1.3.6

2. Use Gauss-Jordan elimination to find a reduced row-echelon form from this augmented matrix. We have already done this in Examples 1.3.6 and 1.3.8 :-

$$\text{RREF :} \quad \left(\begin{array}{rrrrr} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$
$$\quad x_1 \quad x_2 \quad x_3 \quad x_4$$

   This matrix corresponds to a new system of equations:

$$\begin{array}{rcrcrl} x_1 & + & 2x_4 & = & 4 & \quad (A) \\ x_2 & - & x_4 & = & 2 & \quad (B) \\ x_3 & + & x_4 & = & 2 & \quad (C) \end{array}$$

   Remark : The RREF involves 3 leading 1's, one in each of the columns corresponding to the variables x_1, x_2 and x_3. The column corresponding to x_4 contains no leading 1.

   Definition 1.3.10. *The variables whose columns in the RREF contain leading 1's are called* leading variables*. A variable whose column in the RREF does not contain a leading 1 is called a* free variable*.*

   So in this example the leading variables are x_1, x_2 and x_3, and the variable x_4 is free. What does this distinction mean in terms of solutions of the system? The system corresponding to the RREF can be rewritten as follows :

$$\begin{array}{rcrcrl} x_1 & = & 4 & - & 2x_4 & \quad (A) \\ x_2 & = & 2 & + & x_4 & \quad (B) \\ x_3 & = & 2 & - & x_4 & \quad (C) \end{array}$$

   i.e. this RREF tells us how the values of the leading variables x_1, x_2 and x_3 *depend* on that of the free variable x_4 in a solution of the system. In a solution, the free variable x_4 may assume the value of *any* real number. However, once a value for x_4 is chosen, values are immediately assigned to x_1, x_2 and x_3 by equations A, B and C above. For example

(a) Choosing $x_4 = 0$ gives $x_1 = 4 - 2(0) = 4,\ x_2 = 2 + (0) = 2,\ x_3 = 2 - (0) = 2$. Check that $x_1 = 4,\ x_2 = 2,\ x_3 = 2,\ x_4 = 0$ is a solution of the (original) system.

(b) Choosing $x_4 = 3$ gives $x_1 = 4 - 2(3) = -2,\ x_2 = 2 + (3) = 5,\ x_3 = 2 - (3) = -1$. Check that $x_1 = -2,\ x_2 = 5,\ x_3 = -1,\ x_4 = 3$ is a solution of the (original) system.

Different choices of value for x_4 will give different solutions of the system. The number of solutions is *infinite*.

The *general solution* is usually described by the following type of notation. We assign the *parameter* name t to the value of the variable x_4 in a solution (so t may assume any real number as its value). We then have

$$x_1 = 4 - 2t,\ x_2 = 2 + t,\ x_3 = 2 - t,\ x_4 = t;\ t \in \mathbb{R}$$

or

$$\textbf{General Solution :}\ \ (x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t);\ t \in \mathbb{R}$$

This general solution describes the infinitely many solutions of the system : we get a *particular* solution by choosing a specific numerical value for t : this determines specific values for x_1, x_2, x_3 and x_4.

Example 1.3.11. *Solve the following system of linear equations :*

$$\begin{array}{rcrcrcrcrr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \text{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \text{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \text{III} \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & -6 & \text{IV} \end{array}$$

Remark : The first three equations of this system comprise the system of equations of Example 1.3.9. The problem becomes : Can we find a solution of the system of Example 1.3.9 which is in addition a solution of the equation $x_1 - x_2 + x_3 - x_4 = -6$?

SOLUTION We know that every simultaneous solution of the first three equations has the form

$$x_1 = 4 - 2t,\ x_2 = 2 + t,\ x_3 = 2 - t,\ x_4 = t,$$

where t can be any real number . Is there some choice of t for which the solution of the first three equations is also a solution of the fourth? i.e. for which

$$x_1 - x_2 + x_3 - x_4 = -6 \text{ i.e. } (4 - 2t) - (2 + t) + (2 - t) - t = -6$$

Solving for t gives

$$\begin{aligned} 4 - 5t &= -6 \\ \Longrightarrow -5t &= 10 \\ \Longrightarrow t &= 2 \end{aligned}$$

$$t = 2\ :\ x_1 = 4 - 2t = 4 - 2(2) = 0;\ x_2 = 2 + t = 2 + 2 = 4;\ x_3 = 2 - t = 2 - 2 = 0;\ x_4 = t = 2$$

SOLUTION : $x_1 = 0,\ x_2 = 4,\ x_3 = 0,\ x_4 = 2$ (or $(x_1, x_2, x_3, x_4) = (0, 4, 0, 2)$).
This is the *unique* solution to the system in Example 1.3.11.

REMARKS**:**

1. To solve the system of Example 1.3.11 directly (without 1.3.9) we would write down the augmented matrix :

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \\ 1 & -1 & 1 & -1 & -6 \end{pmatrix}$$

Check: Gauss-Jordan elimination gives the reduced row-echelon form :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

which corresponds to the system

$$x_1 = 0; \; x_2 = 4; \; x_3 = 0; \; x_4 = 2$$

i.e. the unique solution is given exactly by the RREF. In this system, all four variables are leading variables. This is always the case for a system which has a unique solution : that each variable is a leading variable, i.e. corresponds in the RREF of the augmented matrix to a column which contains a leading 1.

2. The system of Example 1.3.9, consisting of Equations 1,2 and 3 of that in Example 1.3.11, had an infinite number of solutions. Adding the fourth equation in Example 1.3.11 pinpointed exactly one of these infinitely many solutions.

1.3.4 Consistent and Inconsistent Systems

Example 1.3.12. *Consider the following system :*

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array}$$

To find solutions, obtain a row-echelon form from the augmented matrix :

$$\begin{pmatrix} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{pmatrix} \quad \begin{array}{c} R1 \leftrightarrow R2 \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & -5 & 4 \\ 5 & 3 & -8 & 6 \end{pmatrix}$$

$$\begin{array}{c} R2 \to R2 - 3R1 \\ \longrightarrow \\ R3 \to R3 - 5R1 \end{array} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 2 & 1 \end{pmatrix} \quad \begin{array}{c} R2 \times (-1) \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{c} R3 \to R3 + 2R2 \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{array}{c} R3 \times (-1) \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Row-Echelon Form)

The system of equations corresponding to this REF has as its third equation

$$0x + 0y + 0z = 1 \quad \text{i.e. } 0 = 1$$

This equation clearly has no solutions - no assignment of numerical values to x, y and z will make the value of the expression $0x + 0y + 0z$ equal to anything but zero. Hence the system has no solutions.

Definition 1.3.13. *A system of linear equations is called* inconsistent *if it has no solutions. A system which has a solution is called* consistent.

If a system is inconsistent, a REF obtained from its augmented matrix will include a row of the form 0 0 0 ... 0 1, i.e. will have a leading 1 in its rightmost column. Such a row corresponds to an equation of the form $0x_1 + 0x_2 + \cdots + 0x_n = 1$, which certainly has no solution.

Example 1.3.14.

(a) Find the unique value of t for which the following system has a solution.

$$\begin{array}{rcrcrcrcl} -x_1 & & & + & x_3 & - & x_4 & = & 3 \\ 2x_1 & + & 2x_2 & - & x_3 & - & 7x_4 & = & 1 \\ 4x_1 & - & x_2 & - & 9x_3 & - & 5x_4 & = & t \\ 3x_1 & - & x_2 & - & 8x_3 & - & 6x_4 & = & 1 \end{array}$$

SOLUTION: First write down the augmented matrix and begin Gauss-Jordan elimination.

$$\begin{pmatrix} -1 & 0 & 1 & -1 & 3 \\ 2 & 2 & -1 & -7 & 1 \\ 4 & -1 & -9 & -5 & t \\ 3 & -1 & -8 & -6 & 1 \end{pmatrix} \begin{array}{c} R1 \times (-1) \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 2 & 2 & -1 & -7 & 1 \\ 4 & -1 & -9 & -5 & t \\ 3 & -1 & -8 & -6 & 1 \end{pmatrix}$$

$$\begin{array}{c} R2 \to R2 - 2R1 \\ R3 \to R3 - 4R1 \\ \longrightarrow \\ R4 \to R4 - 3R1 \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & -1 & -5 & -9 & t+12 \\ 0 & -1 & -5 & -9 & 10 \end{pmatrix} \begin{array}{c} R3 \to R3 - R4 \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & 0 & 0 & 0 & t+2 \\ 0 & -1 & -5 & -9 & 10 \end{pmatrix}$$

From the third row of this matrix we can see that the system can be consistent only if $t + 2 = 0$. i.e. only if $t = -2$.

(b) Find the general solution of this system for this value of t.
SOLUTION: Set $t = -2$ and continue with the Gaussian elimination. We omit the third row, which consists fully of zeroes and carries no information.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & -1 & -5 & -9 & 10 \end{pmatrix} \begin{array}{c} R4 \times (-1) \\ \longrightarrow \\ R3 \leftrightarrow R4 \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 1 & 5 & 9 & -10 \\ 0 & 2 & 1 & -9 & 7 \end{pmatrix}$$

$$\begin{array}{c} R3 \to R3 - 2R2 \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 1 & 5 & 9 & -10 \\ 0 & 0 & -9 & -27 & 27 \end{pmatrix} \begin{array}{c} R3 \times (-\frac{1}{9}) \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 1 & 5 & 9 & -10 \\ 0 & 0 & 1 & 3 & -3 \end{pmatrix}$$

$$\begin{array}{c} R1 \to R1 + R3 \\ \longrightarrow \\ R2 \to R2 + 5R3 \end{array} \begin{pmatrix} 1 & 0 & 0 & 4 & -6 \\ 0 & 1 & 0 & -6 & 5 \\ 0 & 0 & 1 & 3 & -3 \end{pmatrix}$$

Having reached a reduced row-echelon form, we can see that the variables x_1, x_2 and x_3 are leading variables, and the variable x_4 is free. We have from the RREF

$$x_1 = -6 - 4x_4, \; x_2 = 5 + 6x_4, \; x_3 = -3 - 3x_4.$$

If we assign the parameter name s to the value of the free variable x_4 in a solution of the system, we can write the general solution as

$$(x_1, x_2, x_3, x_4) = (-6 - 4s, 5 + 6s, -3 - 3s, s), \; s \in \mathbb{R}.$$

Summary of Possible Outcomes when Solving a System of Linear Equations:

1. The system may be inconsistent. This happens if a REF obtained from the augmented matrix has a leading 1 in its rightmost column.
2. The system may be consistent. In this case one of the following occurs :

(a) There may be a unique solution. This will happen if all variables are leading variables, i.e. every column except the rightmost one in a REF obtained from the augmented matrix has a leading 1. In the case the *reduced* row-echelon form obtained from the augmented matrix will have the following form :

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

with possibly some additional rows full of zeroes at the bottom. The unique solution can be read from the right-hand column.

NOTE: If a system of equations has a unique solution, the number of equations must be at least equal to the number of variables (since the augmented matrix must have enough rows to accommodate a leading 1 for every variable).

(b) There may be infinitely many solutions. This happens if the system is consistent but at least one of the variables is free. In this case the number of leading 1s in the row echelon form is less than the number of variables in the system.