Lecture 6: The Reduced Row-Echelon Form (Section 1.3.2)

Definition A matrix is in reduced row-echelon form (RREF) if

1. It is in row-echelon form, and
2. If a particular column contains a leading 1, then *all* other entries of that column are zeros.

If we have a row-echelon form, we can use EROs to obtain a reduced row-echelon form (this is called Gauss-Jordan elimination).

Example To get a RREF from a REF:

$$\begin{bmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R1 \to R1+R2} \begin{bmatrix} 1 & 0 & 3 & 5 & 10 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R2 \to R2-4R3]{R1 \to R1-3R3} \begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

RREF

Remark Different sequences of ERO's on a matrix can lead to different row-echelon forms. However, the reduced row-echelon form of any matrix is unique.

Leading Variables and Free Variables (Section 1.3.3)

Example Find all solutions of the following system :

$$\begin{array}{rcrcrcrcrl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \quad \mathrm{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \quad \mathrm{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \quad \mathrm{III} \end{array}$$

Step 1 Write down the augmented matrix of the system :

$$\begin{array}{c} \left[\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & RHS \end{array} \end{array}$$

Step 2 Use Gauss-Jordan elimination to find a reduced row-echelon form from this augmented matrix.

$$\text{RREF}: \quad \begin{array}{c} \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array} \leftarrow RHS$$

Leading variables and free variables (continued)

This matrix corresponds to a new system of equations:

$$\boxed{\begin{array}{rcrcl} x_1 & + & 2x_4 & = & 4 \\ x_2 & - & x_4 & = & 2 \\ x_3 & + & x_4 & = & 2 \end{array}} \quad \begin{array}{lcrcrcr} (A) & \Longrightarrow & x_1 & = & 4 & - & 2x_4 \\ (B) & \Longrightarrow & x_2 & = & 2 & + & x_4 \\ (C) & \Longrightarrow & x_3 & = & 2 & - & x_4 \end{array} \quad \leftarrow$$

Definition The variables whose columns in the RREF contain leading 1s (x_1, x_2, x_3) are leading variables. A variable whose column in the RREF does not contain a leading 1 (x_4 in this example) is a free variable.

The RREF tells us how the values of the leading variables x_1, x_2 and x_3 depend on that of the free variable x_4 in a solution. The free variable x_4 may assume the value of *any* real number. The number of solutions is *infinite*.

The general solution is described by assigning a parameter name t to the value of each free variable in a solution. In this example

$$\boxed{(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \; t \in \mathbb{R}.}$$

Choosing a value for t gives a particular solution : e.g $t=1$: $(x_1, x_2, x_3, x_4) = (4-2, 2+1, 2-1, 1)$

$= (2, 3, 1, 1)$

Choosing $t = 50$ $\quad x_1 = -96, \ x_2 = 52, \ x_3 = -48, \ x_4 = 50$

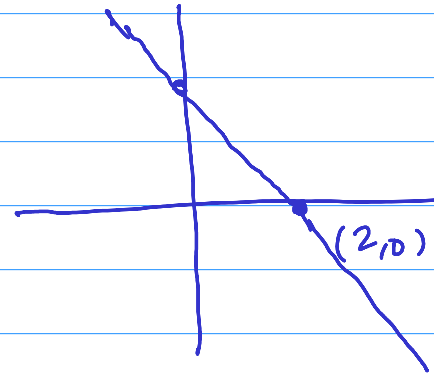
<u>Remark</u> If we think of

$$\boxed{x + y = 2}$$

as a system of equations, its augmented matrix is

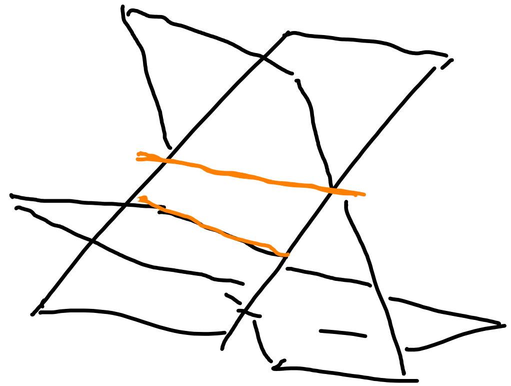
$$\left[\begin{array}{cc|c} 1 & 1 & 2 \end{array}\right]$$

$x = 2 - y$



Example Consider the following system :

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array}$$



To find solutions, obtain a row-echelon form from the augmented matrix :

$$\begin{bmatrix} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & -5 & 4 \\ 5 & 3 & -8 & 6 \end{bmatrix} \xrightarrow[R3 \to R3-5R1]{R2 \to R2-3R1} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R2 \times (-1)} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{R3 \to R3+2R2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right] \xrightarrow{R3 \times (-1)} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Consistent and Inconsistent Systems

The system of equations corresponding to this REF has as its third equation

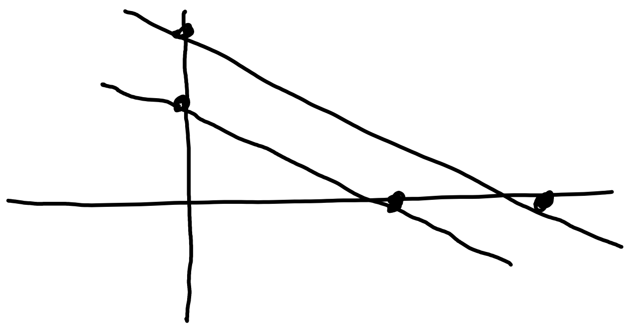
$$0x + 0y + 0z = 1$$

This equation clearly has no solutions - no assignment of numerical values to x, y and z will make the value of the expression $0x + 0y + 0z$ equal to anything but zero. Hence the system has no solution.

Definition A system of linear equations is called inconsistent if it has no solution. A system which has a solution is called *consistent*.

Example

$$x + 2y = 4$$
$$x + 2y = 6$$



Summary of possible outcomes to solving a linear system

1. The system may be inconsistent. This happens if a REF obtained from the augmented matrix has a leading 1 in its rightmost column.
2. The system may be consistent. Then one of the following occurs :
    1. There may be a unique solution. This happens if all variables are leading variables. In the case the RREF has the following form :

    $$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

    with maybe some rows full of zeros at the bottom. The unique solution can be read from the rightmost column.
    2. There may be infinitely many solutions. This happens if the system is consistent but at least one of the variables is free.