Three more concepts from matrix algebra

The $n \times n$ identity matrix For a positive integer n, the $n \times n$ identity matrix, denoted I_n, is the $n \times n$ matrix whose entries in the $(1,1), (2,2), \dots, (n,n)$ positions (the positions on the main diagonal) are all 1, and whose entries in all other positions (all off-diagonal positions) are 0. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The special property that I_n has is that is an *identity element* or *neutral element* for matrix multplication. Multiplying another matrix by it has no effect. This means

- If A is any matrix with n rows, then $I_nA = A$, and
- If B is any matrix with n columns, then $BI_n = B$.
- In particular, if C is a $n \times n$ matrix, then $CI_n = I_nC = C$.

The Inverse of a Matrix

Let A be a square matrix of size $n \times n$. If there exists a $n \times n$ matrix B for which $AB = I_n$ and $BA = I_n$, then A and B are called inverses (or multiplicative inverses) of each other. If it does not already have another name, the inverse of A is denoted A^{-1}.

Example In $M_2(\mathbb{Q})$, the matrices $\begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ -\frac{5}{2} & -\frac{3}{2} \end{pmatrix}$ are inverses of each other.

Not every square matrix has an inverse. For example the 2×2 matrix $\begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix}$ does not.

Exercise Prove that a square matrix can only have one inverse.

The transpose of a matrix

Definition 8

The transpose of the $m \times n$ matrix A, which is denoted A^T, is defined to be the $n \times m$ matrix which has the entries of Row 1 of A in its first column, the entries of Row 2 of A in its second column, and so on.

Example If $A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & 4 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 2 \\ -2 & 0 \\ -3 & 4 \end{pmatrix}$.

For all relevant i and j, the (i, j) entry of A^T is the (j, i) entry of A. If A is $m \times n$, then the products AA^T and $A^T A$ always exist, and they are square matrices of size $m \times m$ and $n \times n$ respectively. Moreover, they are *symmetric*. A square matrix is symmetric if it is equal to its own transpose.

The transpose of a matrix product

Lemma 9

Let A and B be matrices for which the product AB is defined. Then $(AB)^T = B^T A^T$.

The lemma is saying that the transpose of the product A is the product of the transposes of A and B, but in the opposite order.

Proof.

Suppose that the sizes of A and B are $m \times p$ and $p \times n$ respectively. Choose an arbitrary position (i, j) in $(AB)^T$. The entry in this position is

$$\begin{aligned} (AB)^T_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{p} A_{jk} B_{ki} \\ &= \sum_{k=1}^{p} B^T_{ik} A^T_{kj} \end{aligned}$$

Section 1.3: Systems of Linear Equations

Consider the equation

$$2x + y = 3.$$

This is an example of a *linear equation* in the variables x and y. As it stands, the statement "$2x + y = 3$" is neither true nor untrue : it is just a statement involving the abstract symbols x and y. However if we replace x and y with some particular pair of real numbers, the statement will become either true or false.

Definition A pair (x_0, y_0) of real numbers is a solution to the equation $2x + y = 3$ if setting $x = x_0$ and $y = y_0$ makes the equation a true statement.

The set of all solutions to the equation is called its *solution set*.

A collection of linear equations in the same n variables is referred to as a *linear system* or *system of linear equations*. The solution set of the system is the subset of $\mathbb{R}^n$ consisting of those elements that satisfy all of the equations in the system.

Example Solve the linear system

$$\begin{array}{rcrcrl} 2x & + & y & = & 3 & \text{(A)} \\ 4x & + & 3y & = & 4 & \text{(B)} \end{array}$$

We can do this with an "ad hoc" approach. This is harder for a more complicated system with more variables, and/or more equations. We will devise a systematic approach, known as Gauss-Jordan elimination.

Example 10

Find all solutions of the following system :

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$$

We can find solutions by simplifying the system through operations of the following types :

1. We can multiply one equation by a non-zero constant.
2. We can add one equation to another (for example in the hope of eliminating a variable from the result).

We now develop a new technique both for describing our system and for applying operations of the above types more systematically and with greater clarity.

The Augmented Matrix

We associate a *matrix* to our system of equations.

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \leftrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right) \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array}$$

Definition The above matrix is called the *augmented matrix* of the system of equations. We work with the augmented matrix instead directly with the equations, and allow operations of the following types.

1. Multiply a *row* by a non-zero constant.
2. Add a multiple of one row to another row.
3. Interchange two rows in the matrix

Definition Operations of these three types are called Elementary Row Operations (EROs) on a matrix.