Lecture 3: Matrix multplication II

Definition 5

Let A and B be matrices of size $m \times p$ and $p \times n$ respectively. Write $v_1, \dots, v_n$ for the columns of B. Then the product AB is the $m \times n$ matrices whose columns are $Av_1, \dots, Av_n$.

Matrix products are often presented and explained just in terms of their individual entries.

Suppose that A is a $m \times p$ matrix and B is a $p \times n$ matrix, with entries in a field $\mathbb{F}$. The entry in Row i and Column j of A is denoted A_{ij}.

The entry in the the (i, j) position of AB (i.e. Row i and Column j) is the ith entry of the vector Av_j, where the vector v_j is Column j of B.

This is the linear combination of the ith entries of the columns of A (i.e. the entries of Row i of A, with coefficients from Column j of B). It is given by

Entries of Row i of A

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \boxed{\sum_{k=1}^{p} A_{ik}B_{kj}.}$$

entries of Column j of B

Matrix multiplication and the scalar product

We note that the expression for $(\overset{m \times p}{A}\overset{p \times n}{B})_{ij}$ above involves the *scalar product* of two vectors with p entries. For a field $\mathbb{F}$, we write $\mathbb{F}^p$ for the vector space of all vectors with p entries from $\mathbb{F}$.

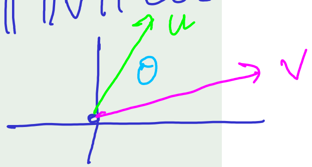
Definition 6

Let $u = (a_1, \dots, a_p)$ and $v = (b_1, \dots, b_p)$ be vectors in $\mathbb{F}^p$. Then the *ordinary scalar product* or *dot product* of u and v is the element of $\mathbb{F}$ defined by

e.g in $\mathbb{R}^2$ $(1,2)\cdot(3,-4) = 1(3)+2(-4) = -5$

$$u \cdot v = a_1 b_1 + a_2 b_2 + \dots + a_p b_p = \sum_{k=1}^{p} a_k b_k.$$

In $\mathbb{R}^2$ $u \cdot v = \|u\|\,\|v\| \cos\theta$



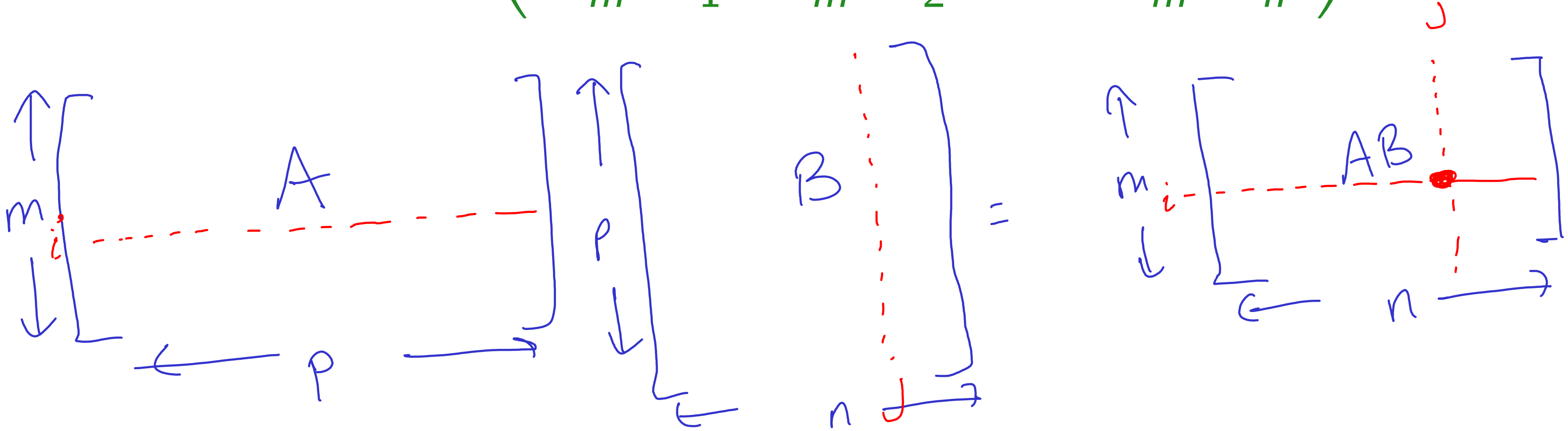
In $\mathbb{R}^2$ $u \cdot v = 0 \iff u \perp v$ (if $u, v \neq (0,0)$)

If $u \cdot v = 0$, we say that $\boxed{u \text{ and } v}$ are *orthogonal* with respect to the scalar product. If $\mathbb{F} = \mathbb{R}$, this means that the vectors u and v are perpendicular in Euclidean space.

Matrix multiplication and the scalar product, II

If A is $m \times p$ with rows $u_1, \dots, u_m$, and B is $p \times n$ with columns $v_1, \dots, v_n$, then the product AB is a table of values of scalar products of Rows of A with Columns of B.

$$AB = \begin{pmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \dots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \dots & u_2 \cdot v_n \\ \vdots & \vdots & & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \dots & u_m \cdot v_n \end{pmatrix}$$



Making sense of matrix multplication

Example Let A be the 3×3 matrix formed by the table that gives the numbers of first year Humanities (H), Engineering (E) and Science (S) students in first year at Eigen University, in 2015, 2016 and 2017.

	H	E	S
2015	50	100	70
2016	60	80	80
2017	80	70	70

$$A = \begin{pmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{pmatrix} \quad 3 \times 3$$

Every fiirst year student at Eigen University takes either Linear Algebra (LA) or Calculus (C) or both. The table below shows the numbers of ECTS credits completed annually in each, by students in each of the three subject areas.

	LA	C
H	10	0
E	15	15
S	20	10

$$B = \boxed{\begin{pmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{pmatrix}} \quad 3 \times 2$$

Now look at the meaning of the entries of the product AB.

$$A = \begin{array}{c} \\ 2015 \\ 2016 \\ 2017 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} H & E & S \end{array} \\ \begin{bmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ H \\ E \\ S \end{array}\!\!\begin{array}{c} \begin{array}{cc} LA & C \end{array} \\ \begin{bmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{bmatrix} \end{array} = B$$

3×3 $\qquad$ 3×2

$$AB = \begin{array}{c} \\ 2015 \\ 2016 \\ 2017 \end{array}\!\!\begin{array}{c} \begin{array}{cc} LA & C \end{array} \\ \begin{bmatrix} \boxed{3400} & 2200 \\ \boxed{3400} & 2000 \\ 3250 & 1750 \end{bmatrix} \end{array}$$

Entry in (i,j) position is the number of credits taken by students in the year that labels Row i of A, in the subject that labels Column j.

Linear Transformations

Linear transformations are the primary functions between vector spaces that are of interest in linear algebra. For now we will stick to linear transformations between spaces of real column vectors.

Definition 7

Let m and n be positive integers. A linear tranformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies

- $T(u + v) = T(u) + T(v)$, and
- $T(\lambda v) = \lambda T(v)$,

for all u and v in $\mathbb{R}^n$, and all scalars $\lambda \in \mathbb{R}$.

addition in $\mathbb{R}^m$

addition in $\mathbb{R}^n$

The Matrix of a Linear Transformation

Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation. Then we can calculate the image under T of any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, if we know the images under T of the *standard basis vectors* $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. From the definition, we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = aT\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + cT\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where A is the 2×3 matrix that has the images of the three standard basis vectors as its three columns.

Matrix multplication is composition

Example $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$ $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$

$T : \mathbb{R}^3 \to \mathbb{R}^2$

Matrix of T: $A = \begin{bmatrix}1 & 1 & -2\\2 & 4 & 3\end{bmatrix}$

$$T\begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{bmatrix}1 & 1 & -2\\2 & 4 & 3\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix} = \begin{bmatrix}-5\\22\end{bmatrix}$$

$T : \mathbb{R}^3 \to \mathbb{R}^2$ Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is another linear transformation, with matrix $B = \begin{bmatrix}1 & -1\\2 & 0\end{bmatrix}$.

$$\mathbb{R}^3 \xrightarrow[T]{} \mathbb{R}^2 \xrightarrow[S]{} \mathbb{R}^2$$

Composition $S \circ T : \mathbb{R}^3 \to \mathbb{R}^2$ is again a linear transformation.

"S after T"

Suppose v is a vector in $\mathbb{R}^3$.

To apply T to v, we calculate Av

To apply S to this, we calculate $B(Av)$

$$(S \circ T)v = S(T(v)) = S(\underset{\in \mathbb{R}^2}{Av}) = B(Av) = BAv$$

BA is the matrix of the composition $S \circ T$, where B and A are respectively the matrices of S and T.

In our example

$$BA = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 2 & 4 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -3 & -5 \\ 2 & 2 & -4 \end{pmatrix}}_{\text{Matrix of } S \circ T}$$