

Section 1.2: Review of Matrix Algebra

A $m \times n$ matrix over a field \mathbb{F} is an array of m rows and n columns, whose entries are elements of \mathbb{F} . (We can take \mathbb{F} to be the field of real numbers.)

The expression $m \times n$ is referred to as the *size* of a matrix (even though what it really describes is the *shape*).

$$\begin{bmatrix} 3 & 1 & 0 & 2 \\ -2 & 0 & 1 & 4 \\ -16 & 1 & -2 & 1006 \end{bmatrix}$$

A 3×4 matrix
over \mathbb{R}

3 rows
4 columns

$M_{m \times n}(\mathbb{F})$ is a vector space ^{its elements can be added together, subtracted, or multiplied by scalars}

The set of all $m \times n$ matrices over \mathbb{F}

e.g. In $M_{2 \times 3}(\mathbb{R})$ $\begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -3 & 0 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 4 & 1 & 7 \end{pmatrix}$

Two matrices can be added together if they have the same size; in this case their sum is obtained by just adding the entries in each position.

$$O_{2 \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The $m \times n$ **zero matrix** is the $m \times n$ matrix whose entries are all zeros. It is the **identity element** for addition of $m \times n$ matrices - this means that addition it to another $m \times n$ matrix has no effect.

The negative of a matrix: $A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$ $-A = \begin{pmatrix} -1 & 3 \\ -4 & -2 \end{pmatrix}$ $A + (-A) = O_{2 \times 2}$

A matrix can be multiplied by a scalar; this means multiplying each of its entries by that scalar. With these operations of addition and scalar multiplication, the set of $m \times n$ matrices over a field \mathbb{F} is a *vector space* over \mathbb{F} .

$$3 \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 6 \\ 3 & -6 & 12 \end{pmatrix}$$

Matrix Multiplication I

We can sometimes also *multiply* matrices.

Definition 2

Suppose that v_1, v_2, \dots, v_k are elements of a vector space V over a field \mathbb{F} . A \mathbb{F} -linear combination of v_1, \dots, v_k is an element of V that has the form $a_1 v_1 + a_2 v_2 + \dots + a_k v_k$, where the a_i are elements of \mathbb{F} . In this situation the a_i are called the *coefficients* in the linear combination.

e.g.
 \mathbb{R}

Definition 3

A *column vector* is a matrix with one column. A *row vector* is a matrix with one row.

a $m \times 1$ matrix

a $1 \times n$ matrix

Definition 4

Let A be a $m \times n$ matrix and let v be a column vector with n entries. Then the matrix-vector product Av is the column vector obtained by taking the linear combination of the columns of A whose coefficients are the entries of v . It is a column vector with m entries.

Example

$$A = \begin{bmatrix} -2 & 1 & 3 & -2 \\ 1 & 0 & 4 & -1 \end{bmatrix} \quad \checkmark \quad = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

2×4 4×1

Matrix-vector product

$$\underline{Av} = 1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

4 columns of A

Matrix multiplication II

Definition 5

Let A and B be matrices of size $m \times p$ and $p \times n$ respectively. Write v_1, \dots, v_n for the columns of B . Then the product AB is the $m \times n$ matrix whose columns are Av_1, \dots, Av_n .

Matrix products are often presented and explained just in terms of their individual entries.

Suppose that A is a $m \times p$ matrix and B is a $p \times n$ matrix, with entries in a field \mathbb{F} . The entry in Row i and Column j of A is denoted A_{ij} .

The entry in the the (i, j) position of AB (i.e. Row i and Column j) is the i th entry of the vector Av_j , where the vector v_j is Column j of B .

This is the linear combination of the i th entries of the columns of A (i.e. the entries of Row i of A , with coefficients from Column j of B). It is given by

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \sum_{k=1}^p A_{ik}B_{kj}.$$