A $m \times n$ matrix over a field $\mathbb{F}$ is an an array of $m$ rows and $n$ columns, whose entries are elements of $\mathbb{F}$. (We can take $\mathbb{F}$ to be the field of real numbers.)

The expression $m \times n$ is referred to as the *size* of a matrix (even though what it really describes is the *shape*).

# $M_{m \times n}(\mathbb{F})$ is a vector space

Two matrices can be added together if they have the same size; in this case their sum is obtained by just adding the entries in each position.

The $m \times n$ zero matrix is the $m \times n$ matrix whose entries are all zeros. It is the identity element for addition of $m \times n$ matrices - this means that addition it to another $m \times n$ matrix has no effect.

A matrix can be multiplied by a scalar; this means multiplying each of its entries by that scalar. With these operations of addition and scalar multiplication, the set of $m \times n$ matrices over a field $\mathbb{F}$ is a *vector space* over $\mathbb{F}$.

# Matrix Multiplication I

We can sometimes also *multiply* matrices.

## Definition 2

Suppose that $v_1, v_2, \dots, v_k$ are elements of a vector space $V$ over a field $\mathbb{F}$. A $\mathbb{F}$-linear combination of $v_1, \dots, v_k$ is an element of $V$ that has the form $a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$, wher the $a_i$ are elements of $\mathbb{F}$. In this situation the $a_i$ are called the *coefficients* in the linear combination.

## Definition 3

A *column vector* is a matrix with one column. A *row vector* is a matrix with one row.

## Definition 4

Let $A$ be a $m \times n$ matrix and let $v$ be a column vector with $n$ entries. Then the matrix-vector product $Av$ is the column vector obtained by taking the linear combination of the columns of $A$ whose coefficients are the entries of $v$. It is a column vector with $m$ entries.

# Matrix multplication II

## Definition 5

Let $A$ and $B$ be matrices of size $m \times p$ and $p \times n$ respectively. Write $v_1, \dots, v_n$ for the columns of $B$. Then the product $AB$ is the $m \times n$ matrices whose columns are $Av_1, \dots, Av_n$.

Matrix products are often presented and explained just in terms of their individual entries.

Suppose that $A$ is a $m \times p$ matrix and $B$ is a $p \times n$ matrix, with entries in a field $\mathbb{F}$. The entry in Row $i$ and Column $j$ of $A$ is denoted $A_{ij}$.

The entry in the the $(i,j)$ position of $AB$ (i.e. Row $i$ and Column $j$) is the $i$th entry of the vector $Av_j$, where the vector $v_j$ is Column $j$ of $B$.

This is the linear combination of the $i$th entries of the columns of $A$ (i.e. the entries of Row $i$ of $A$, with coefficients from Column $j$ of $B$). It is given by

$$(AB)_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \sum_{k=1}^{p} A_{ik}B_{kj}.$$