MA283 Linear Algebra

Semester II 2021-22

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This module is an introduction to the theory, methods, practices and applications of linear algebra.

In linear algebra, the environments that we work in are called *vector spaces*, and the main (but not only) functions of interest are called *linear transformations*.

The entire setup is intricately bound up with the algebra of *matrices*.

The subject and its methods have extraordinary prevalence, importance and applicability in every area of the mathematical sciences.

Vector Spaces and Fields

In order to define a vector space, we need to already have a system of scalars (or numbers) in mind. This set of scalars needs to have the structure of a field. This means that within the set of scalars, it is possible to add, subtract or multiply any two elements, and it is possible to divide any element by any non-zero element, without moving outside the set.

To get a sense of what a field is, we look at some examples and non-examples.

1. The set $\mathbb{R}$ of *real numbers* is a field.
2. The set $\mathbb{R}^+$ of *positive* real numbers is *not* a field. we can't subtract without moving outside $\mathbb{R}^+$ e.g. $3-5 \notin \mathbb{R}^+$
3. The set $\mathbb{Z}$ of integers is *not* a field.
4. The set $\mathbb{Q}$ of rational numbers *is* a field.
5. The set $\mathbb{C}$ of complex numbers *is* a field.
6. For a prime p, the set $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p is a field. We write it as $\mathbb{F}_p$. e.g. $\mathbb{F}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, with addition and multiplication modulo 5.

We can take $\mathbb{R}$ as the "default" example.

3. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Dividing by non-zero elements takes us outside $\mathbb{Z}$
e.g. $1 \in \mathbb{Z}, 3 \in \mathbb{Z}$, but $1/3 \notin \mathbb{Z}$.

Exercise

Which of the following are fields? Why or why not?

1. The set of real numbers of the form $a + b\sqrt{2}$, where a and b are rational.
2. The set of complex numbers of the form $0 + bi$, where $b \in \mathbb{R}$.
3. The set of irrational real numbers.
4. The set of complex numbers of the form $a + bi$, where a and b are rational.

Definition of a Vector Space

Now we come to the most important definition of this course, that of a *vector space*. Informally, a vector space V over a field $\mathbb{F}$ is a set whose elements can be added, subtracted and multiplied by elements of $\mathbb{F}$ (called *scalars*) without ever moving outside the set V.

Example The set $\mathbb{R}^3$ of *column vectors of length 3 over* $\mathbb{R}$ is a vector space over $\mathbb{R}$. Its elements are all vectors of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a, b, c are real numbers. They can be added, subtracted or multiplied by scalars as in these examples.

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, \quad 4\begin{pmatrix} 0 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ -8 \end{pmatrix}$$

$\in \mathbb{R}^3$

2. The set $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences of real numbers.

Addition: add sequences term by term.

If (a_n) and (b_n) are sequences

$a_1, a_2, a_3, \ldots$ $\quad$ $b_1, b_2, b_3, \ldots$

$(a_n)+(b_n)$: $a_1+b_1, a_2+b_2, \ldots$

$(a_n)-(b_n)$: $a_1-b_1, a_2-b_2, \ldots$

Scalar multiplication: just multiply all terms by the scalar.

Remark There is one important difference between $\mathbb{R}^3$ and $\mathbb{R}^n$. In $\mathbb{R}^3$, every element can be expressed as a sum of scalar multiples of $\boxed{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}$

standard basis vectors

e.g. $\begin{pmatrix}\pi\\2\\-6\end{pmatrix} = \pi\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + (-6)\begin{pmatrix}0\\0\\1\end{pmatrix}$.

For $\mathbb{R}^{\mathbb{N}}$, there is no finite set with this property additive inverse

addition in $\mathbb{F}$

addition in V